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GIBBS PHENOMENON AND LEBESGUE CONSTANTS
FOR THE
HAUSDORFF MEANS OF DOUBLE SERIES

by

FRED USTINA

A DISSERTATION
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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a Dissertation entitled "GIBBS PHENOMENON AND LEBESGUE CONSTANTS FOR THE HAUSDORFF MEANS OF DOUBLE SERIES", submitted by FRED USTINA, B. Sc., M. Sc., in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

Since the discovery of Gibbs phenomenon by H . Wilbraham (1848) and the Lebesgue constants by H . Lebesgue (1906), their behavior has been studied extensively under many summability methods in the one dimensional case. However, very little has been done in the two dimensional case. This Dissertation extends this modest beginning in the two dimensional case, comprising a study of the Gibbs phenomenon and Lebesgue constants for the Hausdorff method of summability.

In Chapter 1, by means of a constructive proof, we prove that to investigate the Gibbs phenomenon for an arbitrary function of two variables, of bounded variation in the sense of Hardy-Krause, it is sufficient to investigate it for one such function. Two such functions, suitable for such investigation, are actually given.

In Chapter 2, we apply the Hausdorff means to the Fourier series of one of the two functions constructed in Chapter 1, and give an explicit expression, in terms of an integral involving the Hausdorff weight function in the integrand, for the limit superior of the sequence of Hausdorff transforms of the partial sums of this series at a point of discontinuity.

Chapter 3 deals with the Fourier-Stieltjes transform in two dimensions, and leads to the proof of the result that

$$\int_{1,1}^{n,m} \left| \int_{0,0}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} d^2 g(u,v) \right| ds dt = o(\log m \log n),$$

$$m,n \to \infty \quad ,$$

whenever $g(u,v)$ is of bounded variation and has no mass points bounded away from the coordinate axes.

In Chapter 4, we apply the results of Chapter 3 in an investigation of the Lebesgue constants for the Hausdorff means in two dimensions, and show that these constants are given by

$$c(g) \log m \log n + o(\log m \log n) \; , \qquad m,n \to \infty \quad ,$$

where $c(g)$ is a constant that depends only on the mass points of the Hausdorff weight function $g(u,v)$.

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DEDICATION

This Dissertation is dedicated to the author's wife, Astrid, and to the five little Ustina's, Sephanie Dianne, Melanie Anne, Gregory Ivan, Judith Karen and Barbara Elaine, who sacrificed many of the amenities of life in order to expedite its preparation.

FOREWARD

In the presentation, we follow the practice of numbering the lemmas and theorems for which originality is claimed, and leaving those taken from existing literature unnumbered. Thus

(3.12) Theorem 3

refers to a theorem for which originality is claimed. On the other hand,

(3.1) Theorem

denotes a theorem the proof of which already appears in the existing literature.

If one were to select the highlights of the Dissertation, these would probably include all the results of Chapter 1, Lemma 3 in Chapter 2, and the proofs of Theorems 2, 5 and 6 of Chapter 3, together with the remarks (3.17). The remainder of the material, although equally challenging in many respects, is more of a computational nature.

Although the general approach was motivated by the analogue in the one dimensional case, the work in Chapter 1 may be classed as

original in its entirety. The work in Chapters 2, 3, and 4, although clearly more complex than the corresponding work in the one dimensional case, is patterned after the work of O. Szász [10], A. Zygmund [15], and L. Lorch and D. J. Newman [8], respectively. We have patterned many of their ideas to the two dimensional case, and apologize for the insufficient recognition given them in the sequel.

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CHAPTER 1

Gibbs Phenomenon for Functions of Two Variables

§ 1.

Gibbs phenomenon, which concerns the non-uniform convergence of the partial sums of the Fourier series of a function in the neighborhood of a jump discontinuity, was discovered by H. Wilbraham [13] in 1848. It has been studied very extensively in the case of functions of one variable, and the reader is referred to the author's M. Sc. Thesis [11] for a partial bibliography. In the multi-dimensional case, however, very little appears to have been done. In fact, to our knowledge, there is only one published account of an investigation of Gibbs phenomenon in the multi-dimensional case, that being by Cheng [4] for two dimensions.

In the one dimensional case, it is well known that to study the Gibbs phenomenon for the class of functions of bounded variation which have no removable discontinuities, it is sufficient to study one such function. It is possible that the lack of an analogous result in the two dimensional case has prevented wider investigation of the phenomenon in this case. For the class of functions which are of bounded variation in the sense of Hardy-Krause, we show that the analogous result does hold. We remark that this result obviates the need of some of Cheng's work. We begin with a few preliminaries.

(1.1) Definition

Let $\{\epsilon_n\}$ be a sequence of positive numbers such that $\lim_{n \to \infty} \epsilon_n = o$, and let $\{p_n\}$ be a sequence of natural numbers such that $\lim_{n \to \infty} p_n = \infty$. Let $\{s_n(x)\}$ be a sequence of real valued functions defined for $o < |x-x_o| < \epsilon_n$. Then

$$\limsup_{\substack{n \to \infty \\ x \to x_o}} s_n(x) = \limsup_{k \to \infty} \{s_m(x): m > p_k , \quad o < |x-x_o| < \epsilon_k\}$$

$$\liminf_{\substack{n \to \infty \\ x \to x_o}} s_n(x) = \liminf_{k \to \infty} \{s_m(x): m > p_k , \quad o < |x-x_o| < \epsilon_k\}$$

(1.2) Definition

If the sequence $\{s_n(x)\}$ converges pointwise to a limit function $f(x)$ for $o < |x-x_o| < \epsilon$, then $\{s_n(x)\}$ is said to exhibit the Gibbs phenomenon at $x = x_o$ if one or both of the following inequalities hold.

$$\limsup_{\substack{n \to \infty \\ x \to x_o}} s_n(x) > \limsup_{x \to x_o} f(x)$$

$$\liminf_{\substack{n \to \infty \\ x \to x_o}} s_n(x\ ) < \liminf_{x \to x_o} f(x)$$

(1.3) Definition

By the Gibbs ratio for the sequence $\{s_n(x)\}$, we shall mean the real, positive number defined by

$$\frac{\limsup\limits_{\substack{n \to \infty \\ x \to x_o}} s_n(x) - \liminf\limits_{\substack{n \to \infty \\ x \to x_o}} s_n(x)}{\limsup\limits_{x \to x_o} f(x) - \liminf\limits_{x \to x_o} f(x)}$$

(1.4) Definition

A function $f(x)$ is said to be normalized if, for all x in the domain of definition of $f(x)$, we have

$$f(x) = \frac{1}{2} \{f(x^+) + f(x^-)\} .$$

Let F be the class of all normalized functions of bounded variation. For $f \in F$, let $\{s_n(x)\}$ be the sequence $\{s_n(f;x)\}$, where $s_n(f;x)$ is the n-th partial sum of the Fourier series of $f(x)$. To study the Gibbs phenomenon for an arbitrary $f \in F$, it is sufficient to study it for a specific function in F. For let $f \in F$ have a discontinuity at x_o with $f(x_o^+) - f(x_o^-) = \ell \neq o$, and let

$$\varphi(x) \sim \sum_{k=1}^{\infty} \frac{\sin kx}{k} .$$

Then the function

$$h(x) = f(x) - \frac{\ell}{\pi}\varphi(x-x_o)$$

is continuous at $x = x_o$. The Fourier series corresponding to $h(x)$ must then converge uniformly to $h(x)$ in some neighborhood of x_o, and so its partial sums cannot display the Gibbs phenomenon there. The Gibbs phenomenon for $f(x)$ must then be the same as the Gibbs phenomenon for the function $\frac{\ell}{\pi}\varphi(x-x_o)$, from which it follows that to study the Gibbs phenomenon for $f(x)$, it is sufficient to study it for the function $\varphi(x)$.

We have the corresponding definitions in the two dimensional case.

(1.5) Definition

Let $\{\epsilon_n\}$ and $\{p_n\}$ be the two sequences defined in (1.1), and let $\{s_{m,n}(x,y)\}$ be a sequence of real valued functions of two variables defined for $o < |x-x_o| + |y-y_o| < \epsilon_n$. Then

$$\limsup_{\substack{m,n\to\infty \\ (x,y)\to(x_o,y_o)}} s_{m,n}(x,y)$$

$$= \lim_{k\to\infty} \sup \{s_{m,n}(x,y) \colon m,n > p_k ,$$

$$o < |x-x_o| + |y-y_o| < \epsilon_k\}$$

$$\liminf_{\substack{m,n \to \infty \\ (x,y) \to (x_o,y_o)}} s_{m,n}(x,y)$$

$$= \liminf_{k \to \infty} \{ s_{m,n}(x,y) : m,n > p_k$$

$$o < |x-x_o| + |y-y_o| < \epsilon_k \}$$

(1.6) Definition

If the sequence $\{s_{m,n}(x,y)\}$ converges pointwise to a limit function $f(x,y)$ in the region $o < |x-x_o| + |y-y_o| < \epsilon$, then $\{s_{m,n}(x,y)\}$ is said to exhibit the Gibbs phenomenon at (x_o,y_o) if one or both of the following inequalities hold.

$$\limsup_{\substack{m,n \to \infty \\ (x,y) \to (x_o,y_o)}} s_{m,n}(x,y) > \limsup_{(x,y) \to (x_o,y_o)} f(x,y)$$

$$\liminf_{\substack{m,n \to \infty \\ (x,y) \to (x_o,y_o)}} s_{m,n}(x,y) < \liminf_{(x,y) \to (x_o,y_o)} f(x,y)$$

(1.7) Definition

By the Gibbs ratio for the sequence $\{s_{m,n}(x,y)\}$, we shall mean the real, positive number defined by

$$\frac{\limsup\limits_{\substack{m,n\to\infty \\ (x,y)\to(x_o,y_o)}} s_{m,n}(x,y) - \liminf\limits_{\substack{m,n\to\infty \\ (x,y)\to(x_o,y_o)}} s_{m,n}(x,y)}{\limsup\limits_{(x,y)\to(x_o,y_o)} f(x) - \liminf\limits_{(x,y)\to(x_o,y_o)} f(x)} .$$

In the sequel, we will also need the following:

(1.8) Definition

Let R denote the real line. By a cell $[a_2,b_2;a_1,b_1]$ in $R \times R$, $a_2 \geq a_1$, $b_2 \geq b_1$, we shall mean a rectangular point set, the rectangle being determined by the points (a_2,b_2) , (a_2,b_1) , (a_1,b_2) and (a_1,b_1) . In particular, a cell may be closed $[a_2,b_2;a_1,b_1]$, open $(a_2,b_2;a_1,b_1)$, or half open $(a_2,b_2;a_1,b_1]$ or $[a_2,b_2;a_1,b_1)$; and one or more of the coordinate numbers a_1,b_1,a_2,b_2 may be taken infinite.

(1.9) Definition

By $f(x_2,y_2;x_1,y_1)$ we shall mean the real number defined by

$$f(x_2,y_2) - f(x_2,y_1) - f(x_1,y_2) + f(x_1,y_1) ,$$

where the coordinates are all assumed to be such that $f(x_i,y_j)$, $i,j = 1,2,$ is meaningful. If, for all points (x_2,y_2) , (x_1,y_1) , in the domain of definition of $f(x,y)$, $f(x_2,y_2;x_1,y_1) \geq o$ whenever $x_2 \geq x_1$ and $y_2 \geq y_1$, we shall say that $f(x,y)$ is positively monotonic.

(1.10) Definition

A function $f(x,y)$ is said to be normalized if, for all (x,y) in the domain of definition of $f(x,y)$, we have

$$f(x,y) = \frac{1}{4}\{f(x^+,y^+) + f(x^+,y^-) + f(x^-,y^+) + f(x^-,y^-)\} ;$$

and if one of the coordinates is fixed, say x is fixed, then

$$f(x,y) = \frac{1}{2}\{f(x,y^+) + f(x,y^-)\} .$$

(1.11) Definition

Let $f(x,y)$ be defined on the cell $[a_2,b_2;a_1,b_1]$, $a_1 < a_2$, $b_1 < b_2$, and let $\{x_i\}_o^m$, $\{y_i\}_o^n$ be two sequences such that $a_1 = x_o < x_1 < \ldots < x_m = a_2$, $b_1 = y_o < y_1 < \ldots y_n = b_2$, and form the sum

$$\sum_{i,j=1}^{m,n} |f(x_i,y_j;x_{i-1},y_{j-1})| .$$

If for all such partitions, this sum is uniformly bounded and if M is the least upper bound, and if for some fixed y, resp. fixed x, $f(x,y)$ is a function of bounded variation of the variable x, resp. y, $(x,y) \in [a_2,b_2;a_1,b_1]$, then $f(x,y)$ is said to be of bounded variation on the cell $[a_2,b_2;a_1,b_1]$, and M is its total variation.

(1.12) Remarks

A function which is of bounded variation in the sense of

Definition (1.11) is said to be of bounded variation in the sense of Hardy-Krause. Such a function can always be expressed as the difference of its positive and negative variation functions, and hence as the difference of two non-negative, positively monotonic functions, these functions being positively monotonic in each variable separately. For an account of the many ways of defining the variation of a function of two variables, the reader is referred to Adams [2].

Throughout this dissertation, whenever we speak of the variation of a function, we will mean variation in the sense of Hardy-Krause, although explicit mention of this will often not be made in order to avoid repetition.

Now let F be the class of normalized functions of two variables, periodic in each variable. For $f \in F$, let $\{s_{m,n}(f;x,y)\}$ be the sequence of partial sums of the Fourier series of $f(x,y)$. It is natural to ask, as in the one dimensional case, whether it is sufficient to study the Gibbs phenomenon for a specific function in order to get an understanding of its behavior for an arbitrary $f \in F$. In general, the answer is negative. But for functions of bounded variation in the sense of Hardy-Krause, which have no removable discontinuities, this turns out to be the case:

(1.13) Theorem 1

Let $f(x,y)$ be a normalized function, periodic in each variable, and of bounded variation in the sense of Hardy-Krause in the period

rectangle. The Gibbs phenomenon for $f(x,y)$ at the point $(x,y) = (o,o)$ is the same as the Gibbs phenomenon for the function

$$\psi(f;x,y) = \frac{c}{\pi^2}\ \varphi(x)\cdot\varphi(y) + g_1(o)\ \varphi(x) + g_2(o)\ \varphi(y)$$

where

$$\begin{aligned} \varphi(t) &= 0\ , \quad t = 0 \\ &= \frac{1}{2}\,(\pi - t)\ , \quad 0 < t < 2\pi \\ \varphi(t+2k\pi) &= \varphi(t)\ , \quad k = \pm 1, \pm 2, \pm 3, \ldots \\ c &= f(o^+, o^+; o^-, o^-)\ , \\ g_1(y) &= \frac{1}{\pi}\,\{f(o^+,y) - f(o^-,y)\} - \frac{c}{2\pi}\,\mathrm{sgn}\ y \\ g_2(x) &= \frac{1}{\pi}\,\{f(x,o^+) - f(x,o^-)\} - \frac{c}{2\pi}\,\mathrm{sgn}\ x\ . \end{aligned}$$

(1.14) Corollary

To study the Gibbs phenomenon for an arbitrary function of bounded variation, having no removable discontinuities, it is sufficient to study it for the functions $\varphi(x)$, $\varphi(y)$ and $\varphi(x)\cdot\varphi(y)$.

(1.15) Remarks

By a translation of the axes, it follows that restricting the statement of the theorem to the origin does not restrict the generality of the result. Also, if the difference functions

$$f(x,o^+) - f(x,o^-)$$

$$f(o^+,y) - f(o^-,y)$$

are continuous at the origin, then $c = 0$ and

$$\psi(f;x,y) = g_1(o)\ \varphi(x) + g_2(o)\ \varphi(y).$$

If, in addition, $f(x,y)$ is continuous along one of the two axes, say the x-axis, with the exception of origin, then $g_2(x) \equiv 0$, and

$$\psi(f;x,y) = g_1(o)\ \varphi(x) \ .$$

In both these cases, the Gibbs ratio is easily shown to be

$$\frac{2}{\pi} \int_o^\pi \frac{\sin t}{t} \ dt \ .$$

In general, however, the expression for the Gibbs ratio does not reduce to this simple form.

If the partial sums of the Fourier series of $f(x,y)$ are transformed by a regular, linear summability method, then the transformed sequence will display the Gibbs phenomenon if and only if the transforms of the partial sums of at least one of the functions $\varphi(x)$, $\varphi(y)$, $\varphi(x)\cdot\varphi(y)$ display it. This follows the observation that regardless of the signs of $g_1(o)$, $g_2(o)$ and c , there is always one quadrant in which the terms $g_1(o)\ \varphi(x)$, $g_2(o)\ \varphi(y)$ and $\frac{c}{\pi^2}\ \varphi(x)\ \varphi(y)$ are either all positive or all negative sufficiently near the origin.

In general, for ordinary convergence, the "overshoot" is not of the same magnitude as the "undershoot". For although there is always one quadrant in which the terms comprising $\psi(f;x,y)$ are either all positive or all negative, there are never two quadrants, one in which they are all positive, and another in which they are all negative, unless $c = 0$.

§ 2.

Proof of Theorem 1

We assume that $f(x,y)$ is periodic, of period 2π in each argument, normalized, and of bounded variation. These assumptions imply that $f(x,y)$ has no removable discontinuities, and that the discontinuities, if any, are all along lines parallel to the coordinate axes. If there is one such discontinuity, a translation of the coordinate axes will bring it to the origin, hence there is no loss of generality in restricting the discussion to the origin. We consider the four possibilities.

Case 1 $f(x,y)$ is continuous at the origin. In this case, the Fourier series of $f(x,y)$ converges uniformly to $f(x,y)$ in some neighborhood of the origin, and so its partial sums cannot exhibit the Gibbs phenomenon there. See the corollary (A. 23), Appendix. In this case we have $g_1(o) = g_2(o) = c = 0$, so that by the theorem, the Gibbs phenomenon for $f(x,y)$ is the same as the Gibbs phenomenon for the function $\psi(f;x,y) \equiv 0$.

Case 2 $f(x,y)$ has a discontinuity along one of the coordinate axes. To be definite, assume that the discontinuity is along the y-axis and that, excepting the origin, it is continuous along the x-axis. This would be the case, for instance, if $f(x,y) = \pi - x$ in the period cell $(2\pi, 2\pi; 0, 0)$. The discontinuity of $f(x,y)$ is then such that the difference function

$$\pi g_1(y) = f(o^+,y) - f(o^-,y)$$

is a continuous function of y.

We define the function $\varphi(t)$ as follows:

$$\begin{aligned} \varphi(t) &= -\frac{\pi}{2}, \quad -\pi < t < 0 \\ &= 0, \quad t = 0, \pi \\ &= \frac{\pi}{2}, \quad 0 < t < \pi \end{aligned} \tag{1.16}$$

$$\varphi(t+2k\pi) = \varphi(t), \quad k = \pm 1, \pm 2, \pm 3, \ldots .$$

Then

$$h(x,y) = f(x,y) - \varphi(x)\, g_1(y)$$

is continuous along the y axis, for

$$\begin{aligned} h(o^-,y) &= f(o^-,y) + \pi/2\, g_1(y) \\ &= f(o^-,y) + 1/2\, \{f(o^+,y) - f(o^-,y)\} \\ &= 1/2\, \{f(o^+,y) + f(o^-,y)\} \end{aligned}$$

$$= f(o,y) = h(o,y)$$

$$= f(o^+,y) - 1/2\ \{f(o^+,y) - f(o^-,y)\}$$

$$= f(o^+,y) - \pi/2\ g_1(y)$$

$$= h(o^+,y) \quad .$$

Hence $h(x,y)$ is continuous in some neighborhood of the origin and so the partial sums of the Fourier series of $h(x,y)$ present no Gibbs phenomenon there. Noting that the k-th term of the Fourier series of a finite sum of functions equals the sum of the k-th terms of the Fourier series of the individual functions, it follows that the Gibbs phenomenon for $f(x,y)$ is the same as the Gibbs phenomenon for the function

$$\psi(f;x,y) = \varphi(x)\ g_1(y) \quad . \tag{1.17}$$

Since $h(x,y)$ is continuous along the y axis, the jump of $\psi(f;x,y)$ across the y axis is of the same magnitude and the same sign as that of $f(x,y)$ at every point (o,y). Expanding $\psi(f;x,y)$ in a Fourier series, we have

$$\psi(f;x,y) \sim \sum_{k=1}^{\infty} \frac{\sin\ (2k-1)x}{2k-1} \quad \sum_{-\infty}^{\infty} c_\ell e^{i\ell y}$$

and

$$s_{m,n}(\psi;x,y) = \sum_{k=1}^{m} \frac{\sin (2k-1)x}{2k-1} \sum_{-n}^{n} c_\ell e^{i\ell y} \quad .$$

Then, assuming $g_1(y) \geq 0$, we have

$$\limsup_{\substack{m,n \to \infty \\ (x,y') \to (o,y)}} s_{m,n}(\psi;x,y') = \int_o^\pi \frac{\sin t}{t}\, dt \cdot g_1(y) \; ,$$

where we have used the known result from the one dimensional theory for the first series, and the continuity of the function $g_1(y)$ for the second. The Gibbs ratio is given by

$$(1.18) \qquad \frac{\left\{ \int_o^\pi \frac{\sin t}{t}\, dt - \int_o^{-\pi} \frac{\sin t}{t}\, dt \right\} g_1(y)}{\pi g_1(y)}$$

$$= \frac{2}{\pi} \int_o^\pi \frac{\sin t}{t}\, dt \quad .$$

Hence $f(x,y)$ displays the Gibbs phenomenon along the line of discontinuity $x = 0$, with the Gibbs ratio at every point (o,y) given by (1.18) .

Case 3 $f(x,y)$ is discontinuous along both axes, and the discontinuity is such that the difference functions

$$(1.19) \qquad \pi g_1(y) = f(o^+,y) - f(o^-,y)$$

$$\pi g_2(x) = f(x,o^+) - f(x,o^-)$$

are both continuous. This is the case, for instance, if $f(x,y) = 2\pi - (x+y)$ in the period cell $(2\pi,2\pi;\ 0,0)$. In this case we consider the function

$$h(x,y) = f(x,y) - \varphi(x)\, g_1(y) - \varphi(y)\, g_2(x) ,$$

where $\varphi(t)$ is defined by (1.6). To show that $h(x,y)$ is continuous along the coordinate axes, we have

$$\begin{aligned} h(x,o^+) - h(x,o^-) &= f(x,o^+) - f(x,o^-) - \varphi(x)\,\{g_1(o^+)-g_1(o^-)\} \\ &\quad - \{\varphi(o^+) - \varphi(o^-)\}\, g_2(x) \\ &= f(x,o^+) - f(x,o^-) - \pi g_2(x) \\ &= 0 \end{aligned}$$

by (1.19). Hence $h(x,o^+) = h(x,o^-) = h(x,o)$, where again the last equality follows from the observation that $h(x,y)$ is a sum of normalized functions, and is hence normalized itself. Similarly,

$$h(o^+,y) = h(o^-,y) = h(o,y) \quad .$$

In particular, we prove that $h(x,y)$ is continuous at the origin. We have

$$\begin{aligned} h(o^+,o^+) &= f(o^+,o^+) - \tfrac{\pi}{2}g_1(o^+) - \tfrac{\pi}{2}g_2(o^+) \\ &= f(o^+,o^+) - \tfrac{1}{2}\,\{f(o^+,o^+) - f(o^-,o^+)\} - \tfrac{1}{2}\{f(o^+,o^+) - f(o^+,o^-)\} \\ &= \tfrac{1}{2}\,\{f(o^+,o^-) + f(o^-,o^+)\} \\ &= f(o^-,o^-) + \tfrac{\pi}{2}g_1(0^-) + \tfrac{\pi}{2}g_2(o^-) \\ &= h(o^-,o^-) \quad . \end{aligned}$$

$$\begin{aligned}
h(o^+,o^-) &= f(o^+,o^-) - \tfrac{\pi}{2}g_1(o^-) + \tfrac{\pi}{2}g_2(o^+) \\
&= f(o^+,o^-) - \tfrac{1}{2}\{f(o^+,o^-) - f(o^-,o^-)\} \\
&\qquad + \tfrac{1}{2}\{f(o^+,o^+) - f(o^+,o^-)\} \\
&= \tfrac{1}{2}\{f(o^+,o^+) + f(o^-,o^-)\} \\
&= f(o^-,o^+) + \tfrac{1}{2}\{f(o^+,o^+) - f(o^-,o^+)\} \\
&\qquad - \tfrac{1}{2}\{f(o^-,o^+) - f(o^-,o^-)\} \\
&= f(o^-,o^+) + \tfrac{\pi}{2}g_1(o^+) - \tfrac{\pi}{2}g_2(o^-) \\
&= h(o^-,o^+) \quad .
\end{aligned}$$

Then

$$\begin{aligned}
h(o^+,o^+) - h(o^+,o^-) &= \tfrac{1}{2}\{f(o^+,o^-) + f(o^-,o^+)\} \\
&\qquad - \tfrac{1}{2}\{f(o^+,o^+) + f(o^-,o^-)\} \\
&= \tfrac{1}{2}\{f(o^+,o^-)-f(o^+,o^+)\} + \tfrac{1}{2}\{f(o^-,o^+) - f(o^-,o^-)\} \\
&= -\tfrac{\pi}{2}g_2(o^+) + \tfrac{\pi}{2}g_2(o^-) \\
&= 0
\end{aligned}$$

where we have used the continuity of $g_2(x)$. Then $h(o^+,o^+) = h(o^+,o^-)$ $= h(o^-,o^+) = h(o^-,o^-) = h(o,o)$, where the last equality follows by the normality of $h(x,y)$. Then also $h(o,o) = h(o,o^+) = h(o,o^-)$ $= h(o^+,o) = h(o^-,o)$, from which the continuity of $h(x,y)$ at the

origin follows. Hence the Fourier series of $h(x,y)$ present no Gibbs phenomenon there, so that the Gibbs phenomenon for $f(x,y)$ is the same as for the function

$$\psi(f;x,y) = \varphi(x)\, g_1(y) + \varphi(y)\, g_2(x) \quad . \tag{1.20}$$

Then

$$\psi(f;x,y) \sim \sum_{k=1}^{\infty} \frac{\sin\,(2k-1)x}{2k-1} \sum_{-\infty}^{\infty} c_\ell e^{i\ell y} + \sum_{k=1}^{\infty} \frac{\sin\,(2k-1)y}{2k-1} \sum_{-\infty}^{\infty} d_\ell e^{i\ell x}$$

and

$$s_{mn}(\psi;x,y) = \sum_{k=1}^{m} \frac{\sin\,(2k-1)x}{2k-1} \sum_{-n}^{n} c_\ell e^{i\ell y} + \sum_{k=1}^{n} \frac{\sin\,(2k-1)y}{2k-1} \sum_{-m}^{m} d_\ell e^{i\ell x} \quad .$$

At the origin, the Gibbs ratio for the function $\psi(f;x,y)$, and hence also for the function $f(x,y)$, is given by

$$\frac{2\int_0^\pi \frac{\sin t}{t}\,dt\,\{g_1(0)\ \mathrm{sgn}\ g_1(0)+g_2(0)\ \mathrm{sgn}\ g_2(0)\}}{\pi\{g_1(0)\ \mathrm{sgn}\ g_1(0)+g_2(0)\ \mathrm{sgn}\ g_2(0)\}} \tag{1.21}$$

$$= \frac{2}{\pi}\int_0^\pi \frac{\sin t}{t}\,dt \quad ,$$

where again we have used the continuity of the functions $g_1(y)$ and $g_2(x)$, and the known behavior of the partial sums of the Fourier series of $\varphi(t)$.

Case 4 $f(x,y)$ is discontinuous along the coordinate axes, and the normalized difference functions $g_1(y)$, $g_2(x)$, defined by (1.19) are both discontinuous at the origin. This is the case, for instance, if we take $f(x,y) = (\pi-x)(\pi-y)$ in the period cell $(2\pi,2\pi; 0,0)$.

We observe that $g_1(y)$ and $g_2(x)$ are either both continuous, or both discontinuous at the origin, for

$$(1.22) \qquad \pi g_1(o^+) - \pi g_1(o^-) = f(o^+,o^+) - f(o^-,o^+) - f(o^+,o^-) + f(o^-,o^-)$$

$$= f(o^+,o^+; o^-,o^-)$$

$$= \pi g_2(o^+) - \pi g_2(o^-) \quad .$$

Hence the present case is the only remaining case to consider.

As before, our aim is to give the construction of a new function, in some sense simpler than the original function $f(x,y)$, such that this new function will exhibit the same Gibbs phenomenon at the origin as $f(x,y)$, and such that the discontinuous components of this new function will be specified in terms of known, discontinuous functions. To this end, we start with the definition of $g_1(y)$ and $g_2(x)$, as given by (1.19), noting that now these functions are discontinuous at the origin, the discontinuity consisting of a simple jump in the amount of $\frac{1}{\pi} f(o^+,o^+; o^-,o^-)$, by (1.22).

We decompose $g_1(y)$ and $g_2(x)$ into continuous and discontinuous components as follows:

(1.23) $$g_1(y) = g_1'(y) + \varphi(y) \frac{c}{\pi^2}$$

$$g_2(x) = g_2'(x) + \varphi(x) \frac{c}{\pi^2}$$

where $g_1'(y)$ and $g_2'(x)$ are now continuous at the origin, and $\varphi(t)$ is defined by (1.16). Now as a normalized function,

(1.24) $$h_1(x,y) = f(x,y) - \varphi(y)\, g_2(x)$$

is continuous in some neighborhood of (o,o), except along the y axis. To see this, we note that the continuity of $h_1(x,y)$ in each of the four quadrants, excluding the axes, is implied by the assumed conditions on $f(x,y)$ and the obvious continuity of $\varphi(y)\, g_2(x)$. Along the x axis we have

$$\begin{aligned} h_1(x,o^+) - h_1(x,o^-) &= f(x,o^+) - f(x,o^-) - \pi g_2(x) \\ &= 0 \end{aligned}$$

by (1.19), so that $h_1(x,y)$ is continuous along the x axis.

Next, we consider the difference function $g_1(h_1;y)$, where

(1.25) $$\pi g_1(h_1;y) = h_1(o^+,y) - h_1(0^-,y) \quad .$$

This function is continuous at the origin, and so as a normalized function,

$$h(x,y) = h_1(x,y) - \varphi(x)\, g_1(h_1;y)$$

is continuous along the y axis, and so also in some neighborhood of the origin. We shall prove this after putting $h(x,y)$ into a more convenient form. By (1.23), (1.24) and (1.25) we have

$$\begin{aligned} h(x,y) &= f(x,y) - \varphi(y)\, g_2(x) - \frac{\varphi(x)}{\pi} \{f(o^+,y) - \varphi(y)\, g_2(o^+) \\ &\qquad - f(o^-,y) + \varphi(y)\, g_2(o^-)\} \\ &= f(x,y) - \varphi(y)\, g_2'(x) - \varphi(y)\, \varphi(x) \frac{c}{\pi^2} - \varphi(x)\, g_1'(y) \\ &\qquad - \varphi(x)\, \varphi(y) \frac{c}{\pi^2} + \varphi(x)\, \varphi(y) \cdot \frac{1}{\pi} \{g_2(o^+) - g_2(o^-)\} \\ &= f(x,y) - \varphi(y)\, g_2'(x) - \varphi(x)\, g_1'(y) - \varphi(x)\, \varphi(y) \frac{c}{\pi^2} , \end{aligned}$$

where

$$c = g(o^+,o^+;\ o^-,o^-) \quad .$$

We prove that $h(x,y)$ is continuous in some neighborhood of the origin. Its continuity in each of the four quadrants, excluding the coordinate axes, is clear. Along the x axis, we have

$$\begin{aligned} h(x,o^+) - h(x,o^-) &= f(x,o^+) - f(x,o^-) - (\tfrac{\pi}{2} + \tfrac{\pi}{2})\, g_2'(x) \\ &\qquad - \varphi(x)\, \{g_1'(o^+) - g_2'(o^-)\} - \varphi(x)\, \frac{c}{\pi} \\ &= \pi g_2(x) - \pi g_2'(x) - \pi\varphi(x) \frac{c}{\pi^2} \end{aligned}$$

by (1.19) and (1.23). Hence, using also the normality of $h(x,y)$, we have $h(x,o^+) = h(x,o^-) = h(x,o)$. A similar calculation shows that

$h(o^+,y) = h(o^-,y) = h(o,y)$.

For oblique approach to the origin, we have

$$
\begin{aligned}
h(o^+,o^+) &= f(o^+,o^+) - \frac{\pi}{2}\, g_2'(o^+) - \frac{\pi}{2}\, g_1'(o^+) - \frac{c}{4} \\
&= f(o^+,o^+) - \frac{\pi}{2}\, \{g_2(o^+) - \frac{\pi}{2}\,\frac{c}{\pi^2}\, \} \\
&\qquad - \frac{\pi}{2}\, \{g_1(o^+) - \frac{\pi}{2}\,\frac{c}{\pi^2}\, \} - \frac{c}{4} \\
&= f(o^+,o^+) - \frac{1}{2}\, \{f(o^+,o^+) - f(o^+,o^-) + f(o^+,o^+) \\
&\qquad - f(o^-,o^+)\} + \frac{c}{4} \\
&= \frac{1}{2}\, \{f(o^+,o^-) + f(o^-,o^+)\} + \frac{1}{4}\, \{f(o^+,o^+) - f(o^-,o^+) \\
&\qquad - f(o^+,o^-) + f(o^-,O^-)\} \\
&= \frac{1}{4}\, \{f(o^+,o^+) + f(o^+,o^-) + f(o^-,o^+) + f(o^-,o^-)\} \\
&= f(o,o)
\end{aligned}
$$

$$
\begin{aligned}
h(o^+,o^-) &= f(o^+,o^-) + \frac{\pi}{2}\, g_2'(o^+) - \frac{\pi}{2}\, g_1'(o^-) + \frac{c}{4} \\
&= f(o^+,o^-) + \frac{\pi}{2}\, \{g_2(o^+) - \frac{\pi}{2}\,\frac{c}{\pi^2}\, \} \\
&\qquad - \frac{\pi}{2}\, \{g_1(o^-) + \frac{\pi}{2}\,\frac{c}{\pi^2}\, \} + \frac{c}{4} \\
&= f(o^+,o^-) + \frac{1}{2}\, \{f(o^+,o^+) - f(o^+,o^-)\} \\
&\qquad - \frac{1}{2}\, \{f(o^+,o^-) - f(o^-,o^-)\} - \frac{c}{4}
\end{aligned}
$$

$$= \frac{1}{2}\{f(o^+,o^+) + f(o^-,o^-)\} - \frac{1}{4}\{f(o^+,o^+)$$

$$- f(o^+,o^-) - f(o^-,o^+) + f(o^-,o^-)\}$$

$$= \frac{1}{4}\{f(o^+,o^+) + f(o^+,o^-) + f(o^-,o^+) + f(o^-,o^-)\}$$

$$= f(o,o)$$

$$h(o^-,o^+) = f(o^-,o^+) - \frac{\pi}{2} g_2'(o^+) + \frac{\pi}{2} g_1'(o^-) + \frac{c}{4}$$

$$= f(o^-,o^+) - \frac{\pi}{2}\{g_2(o^-) + \frac{\pi}{2}\frac{c}{\pi^2}\}$$

$$+ \frac{\pi}{2}\{g_1(o^+) - \frac{\pi}{2}\frac{c}{\pi^2}\} + \frac{c}{4}$$

$$= f(o^-,o^+) - \frac{1}{2}\{f(o^-,o^+) - f(o^-,o^-)\}$$

$$+ \frac{1}{2}\{f(o^+,o^+) - f(o^-,o^+)\} - \frac{c}{4}$$

$$= \frac{1}{2}\{f(o^+,o^+) + f(o^-,o^-)\} - \frac{1}{4}\{f(o^+,o^+) - f(o^+,o^-)$$

$$- f(o^-,o^+) + f(o^-,o^-)\}$$

$$= \frac{1}{4}\{f(o^+,o^+) + f(o^+,o^-) + f(o^-,o^+) + f(o^-,o^-)\}$$

$$= f(o,o)$$

$$h(o^-,o^-) = f(o^-,o^-) + \frac{\pi}{2} g_2'(o^-) + \frac{\pi}{2} g_1'(o^-) - \frac{c}{4}$$

$$= f(o^-,o^-) + \frac{\pi}{2}\{g_2(o^-) + \frac{\pi}{2}\frac{c}{\pi^2}\} + \frac{\pi}{2}\{g_1(o^-) + \frac{\pi}{2}\frac{c}{\pi^2}\} - \frac{c}{4}$$

$$= f(o^-,o^-) + \frac{1}{2}\{f(o^-,o^+) - f(o^-,o^-)\}$$

$$+ \frac{1}{2}\{f(o^+,o^-) - f(o^-,o^-)\} + \frac{c}{4}$$

$$= \frac{1}{2}\{f(o^-,o^+) + f(o^+,o^-)\} + \frac{1}{4}\{f(o^+,o^+) - f(o^+,o^-)$$

$$- f(o^-,o^+) + f(o^-,o^-)\}$$

$$= \frac{1}{4}\{f(o^+,o^+) + f(o^+,o^-) + f(o^-,o^+) + f(o^-,o^-)\}$$

$$= f(o,o) \quad .$$

Hence $h(x,y)$, as a normalized function, is continuous at the origin, and its Fourier series can show no Gibbs phenomenon there. The Gibbs phenomenon for $f(x,y)$ must then be the same as for the function

$$\psi(f;x,y) = \varphi(y)\, g_2'(x) + \varphi(x)\, g_1'(y) + \varphi(x)\, \varphi(y) \frac{c}{\pi^2} \quad . \tag{1.26}$$

Now let $f_1(x,y)$ be a function such that for the oblique approach to the origin, we have

$$f_1(o^{\pm},o^{\pm}) = f(o^{\pm},o^{\pm}) \quad ,$$

and such that the continuous components of its difference functions (1.19) are constant:

$$g_2'(f_1;x) = g_2'(f_1;o) = g_2'(o)$$

$$g_1'(f_1;y) = g_1'(f_1;o) = g_1'(o) \quad .$$

It is clear that $f_1(o^+,o^+;\ o^-,o^-) = f(o^+,o^+;\ o^-,o^-) = c$. We let

$$\psi(f;x,y) - \psi(f_1;x,y) = \varphi(y)\ \{g_2'(x) - g_2'(o)\}$$
$$+ \varphi(x)\ \{g_1'(y) - g_1'(o)\} \ .$$

This difference function is continuous at the origin, this continuity being implied by the continuity of the functions $g_2'(x)$ and $g_1'(y)$ there. Hence the Gibbs phenomenon for $\psi(f_1;x,y)$ must be the same as the Gibbs phenomenon for $\psi(f;x,y)$, and so the same as for the function $f(x,y)$.

Finally, we define $\varphi_1(t)$ as follows:

$$\begin{aligned} \varphi_1(t) &= \tfrac{1}{2}(\pi-t) \ , && 0 < t < 2\pi \\ &= 0 \ , && t = 0 \\ \varphi_1(t+2k\pi) &= \varphi_1(t) \ , && k = \pm 1, \pm 2, \pm 3, \ldots \ . \end{aligned}$$

Let

$$\psi_1(f_1;x,y) = \varphi_1(y)\ g_2'(o) + \varphi_1(x)\ g_1'(o) + \varphi_1(x)\ \varphi_1(y)\ \frac{c}{\pi^2} \ , \tag{1.27}$$

and form the difference function

$$\psi_1(f_1;x,y) - \psi(f_1;x,y) = g_2'(o)\ \{\varphi_1(y) - \varphi(y)\}$$
$$+ g_1'(o)\ \{\varphi_1(x) - \varphi(x)\} + \frac{c}{\pi^2}\ \{\varphi_1(x)\ \varphi_1(y) - \varphi(x)\ \varphi(y)\} \ .$$

This difference function is again continuous at the origin, and so the

Gibbs phenomenon for $\psi_1(f_1;x,y)$ there must be the same as the Gibbs phenomenon for $f(x,y)$.

We complete the proof of Theorem 1 by dropping the primes and subscripts in (1.27), except for the subscripts on $g_2'(o)$ and $g_1'(o)$, and noting that

$$g_1'(y) = g_1(y) - \varphi(y)\,\frac{c}{\pi^2}$$

$$= \frac{1}{\pi}\,\{f(o^+,y) - f(o^-,y)\} - \frac{c}{2\pi}\ \mathrm{sgn}\ y$$

$$g_2'(x) = g_2(x) - \varphi(x)\,\frac{c}{\pi^2}$$

$$= \frac{1}{\pi}\,\{f(x,o^+) - f(x,o^-)\} - \frac{c}{2\pi}\ \mathrm{sgn}\ x \quad .$$

(1.28) Remarks

We observe that if the difference functions (1.19) are continuous at the origin, then $c = 0$ and (1.26) reduces to (1.20). If, in addition, one of the two difference functions, say $g_2(x)$, is identically zero in some neighborhood of the origin, then in that neighborhood (1.20) reduces to (1.17); and if both differences functions are identically zero, then (1.17) reduces to zero identically in some neighborhood of the origin. Hence our results are consistent.

CHAPTER 2

Gibbs Phenomenon for the Hausdorff Means of Double Series

§ 1.

Using the known results from the one dimensional theory, by the corollary (1.14), it is clear that for ordinary convergence, a function $f(x,y)$, of bounded variation and having no removable discontinuities, will always exhibit the Gibbs phenomenon at a point of discontinuity. If, now, we express $f(x,y)$ by its Fourier series, form the sequence $\{s_{m,n}(f;x,y)\}$ of partial sums of this series, and then apply a summability method to this sequence, it is natural to ask whether the transformed sequence will also exhibit the Gibb phenomenon.

For the one dimensional case, this question has been answered for a large number of regular summability methods. For the two dimensional case, to our knowledge this question has been studied only for the circular Riesz means. The interested reader is referred to Cheng [4]. We remark that Cheng's investigation was under a somewhat more restricted hypothesis than that now allowed by Theorem 1 of Chapter 1, in the sense that he limited his investigation to functions which factorize:

$$f(x,y) = f_1(x) \cdot f_2(y) \quad .$$

In this chapter, we establish some preliminary results pertaining to the study of Gibbs phenomenon for the Hausdorff means of double series. The work is a natural extension of the work of Szász [10] for the one dimensional case. We use some of his results, and follow him in the manner of approach.

(2.1) Definition

A summability method is said to be regular if it sums every convergent sequence to its proper sum. It is said to be totally regular if, in addition, the transformed sequence diverges to infinity whenever the original sequence does.

(2.2) Definition

Let $\{s_m\}$ be a given sequence. The Hausdorff transform of this sequence is the sequence $\{h_m\}$, where

$$h_m = \sum_{k=o}^{m} \binom{m}{k} s_k \int_o^1 u^k (1-u)^{m-k} dg(u) \quad .$$

The transform is regular if and only if $g(o) = g(o^+)$, $g(1) - g(o) = 1$, and $g(u)$ is of bounded variation.

(2.3) Remarks

For convenience, we take $g(o) = 0$, so that $g(1) = 1$.

For a detailed discussion of the Hausdorff transform in the one dimensional case, the reader is referred to Widder [12].

As we have seen earlier, to investigate the Gibbs phenomenon for an arbitrary, normalized function of bounded variation in the one dimensional case, it is sufficient to investigate it for one such function. We consider the function

$$\begin{aligned}(2.4) \qquad \psi(x) &= \frac{1}{2}(\pi - x), && 0 < x < 2\pi \\ &= 0, && c = 0 \\ \psi(x+2k\pi) &= \psi(x), && k = \pm 1, \pm 2, \pm 3, \ldots,\end{aligned}$$

so that

$$\psi(x) \sim \sum_{k=1}^{\infty} \frac{\sin kx}{k}$$

and

$$s_m(\psi;x) = \sum_{k=1}^{m} \frac{\sin kx}{k}$$

We observe that $\psi(x)$ and $s_m(\psi;x)$ are odd, periodic functions of period 2π, and hence to investigate the Gibbs phenomenon for $\psi(x)$, it is sufficient to investigate it in the interval $0 \leq x < \pi$.

For the Gibbs phenomenon for the Hausdorff means of the Fourier series of $\psi(x)$, Szász [10] obtained the following results:

(2.5) Theorem

Let $h_m(\psi;x)$ be the m-th Hausdorff transform of the sequence $\{s_m(\psi;x)\}$. If $x_m \to 0$ and $mx_m \to \tau \leq \infty$

$$h_m(\psi;x_m) \to \int_0^1 \int_0^\tau \frac{\sin su}{s} \, ds dg(u)$$

(2.6) Theorem

For the Hausdorff means $h_m(\psi;x)$,

$$\limsup_{\substack{m \to \infty \\ x \to 0}} h_m(\psi;x) = \max_{\tau > 0} \int_0^1 \{1-g(u)\} \frac{\sin \tau u}{u} \, du \, .$$

If this maximum is attained for $\tau = \tau_0$, then

$$\limsup_{\substack{m \to \infty \\ x \to x_0}} h_m(\psi;x) = \lim_{mx_m \to \tau_0} h_m(\psi;x_m) \, .$$

We seek to extend these results to the two dimensional case.

(2.7) Definition

Let $\{s_{m,n}\}$ be a given double sequence. The Hausdorff transform of this sequence is the sequence $\{h_{m,m}\}$, where

$$h_{m,n} = \sum_{k,\ell=0}^{m\ n} \binom{n}{\ell}\binom{m}{k} s_{k\ell} \int_{0,0}^{1,1} u^k(1-u)^{m-k} \, v^\ell(1-v)^{n-\ell} d^2g(u,v) \, .$$

The transform is regular if and only if $g(o,v) = g(o^+,v) = g(u,o) = g(u,o^+)$, $g(1,1) - g(o,o) = 1$, and $g(u,v)$ is of bounded variation.

(2.8) Remarks

As in the earlier case, for convenience we take $g(o,o) = 0$, so that $g(1,1) = 1$. We give a brief, expository development of some of the theory of the Hausdorff means for double sequences in the Appendix. For the original developement of these means, the reader is referred to Adams [1] and Hallenbach [5].

By the corollary to Theorem 1 of Chapter 1, to study the Gibbs phenomenon for an arbitrary, normalized function of bounded variation in the two dimensional case, it is sufficient to study it for the functions $\psi(x)$, $\psi(y)$ and $\varphi(x,y) = \psi(x)\,\psi(y)$, where $\psi(x)$ is defined by (2.4) . As before, it is sufficient to investigate the Gibbs phenomenom for $0 \leq x,y < \pi$. We will assume these restrictions on the variables x,y without further explicit mention.

We also observe that if $g(u,v)$ factorizes, so that $g(u,v) = g_1(u)\,g_2(v)$, then the transform for the function $\varphi(x,y)$ breaks down into a product of regular one dimensional transforms. Hence we will assume that no such factorization exists.

We first examine the double Hausdorff means of the functions $\psi(x)$, $\psi(y)$.

(2.9) Theorem 1

If $h_{m,n}(\psi;x)$, $h_{m,n}(\psi;y)$ denote the mn-th regular Hausdorff transforms of the functions $\psi(x)$, resp. $\psi(y)$, corresponding to the weight function $g(u,v)$, then

$$h_{m,n}(\psi;x) = h_m(\psi;x)$$

$$h_{m,n}(\psi;y) = h_n(\psi;y)$$

where $h_m(\psi;x)$, $h_n(\psi;y)$ denote the m-th , resp. n-th regular, one dimensional Hausdorff transforms corresponding to the weight functions $g(u,1)$, resp . $g(1,v)$.

Proof

$$(2.10) \qquad \psi(x) \sim \sum_{k=1}^{\infty} \frac{\sin kx}{k}$$

so that

$$(2.11) \qquad s_m(\psi;x) = \sum_{k=1}^{m} \frac{\sin kx}{k}$$

$$= \frac{1}{2} x + \int_0^x \frac{\sin (m+\frac{1}{2})s}{2 \sin \frac{1}{2} s} \, ds \quad .$$

Then

$$h_{m,n}(\psi;x) = \sum_{k,\ell=o}^{m,n} \binom{n}{\ell}\binom{m}{k} s_k(\psi;x) \int_{o,o}^{1,1} u^k(1-u)^{m-k} v^{\ell}(1-v)^{n-\ell} d^2 g(u,v) \tag{2.12}$$

$$= \sum_{k,\ell=o}^{m,n} \binom{n}{\ell}\binom{m}{k} \{-\frac{1}{2}x + \frac{1}{2}\int_o^x \frac{\sin\,(k+\frac{1}{2})s}{\sin\frac{1}{2}\,s}\, ds\} \cdot$$

$$\cdot \int_{o,o}^{1,1} u^k(1-u)^{m-k} v^{\ell}(1-v)^{n-\ell} d^2 g(u,v)$$

$$= -\frac{1}{2}\,x + \frac{1}{2} \sum_{k=o}^{m} \binom{m}{k} \int_o^x \frac{\sin\,(k+\frac{1}{2})s}{\sin\frac{1}{2}\,s}\, ds \cdot$$

$$\cdot \int_o^1 u^k(1-u)^{m-k} \int_o^1 d^2 g(u,v)$$

$$= -\frac{1}{2}\,x + \frac{1}{2} \sum_{k=o}^{m} \binom{m}{k} \int_o^x \frac{\sin\,(k+\frac{1}{2})\,s}{\sin\frac{1}{2}\,s}\, ds \cdot$$

$$\cdot \int_o^1 u^k(1-u)^{m-k} dg(u,1)$$

$$= h_m(\psi;x) \quad .$$

Here we have used the fact that the Hausdorff transform corresponding to the weight function $g(u,v)$ is regular, so that $dg(u,o) \equiv 0$ and $g(1,1) = 1$, so that, being of bounded variation, $g(u,1)$ is a regular Hausdorff weight function for the one dimensional transform. This proves half of the theorem. The other half is proved by a parallel argument.

We now turn to examine the Hausdorff transform of the sequence of partial sums of the Fourier series of $\varphi(x,y)$. We have

$$(2.13) \qquad \varphi(x,y) \sim \sum_{k=1}^{\infty} \frac{\sin kx}{k} \sum_{\ell=1}^{\infty} \frac{\sin \ell y}{\ell} ,$$

so that

$$(2.14) \qquad s_{m,n}(\varphi;x,y) = \sum_{k=1}^{m} \frac{\sin kx}{k} \sum_{\ell=1}^{n} \frac{\sin \ell y}{\ell}$$

$$= \frac{1}{4} \int_{o,o}^{y,x} \left\{-1 + \frac{\sin (m+\frac{1}{2})s}{\sin \frac{1}{2} s}\right\} \left\{-1 + \frac{\sin (n+\frac{1}{2})t}{\sin \frac{1}{2} t}\right\} dsdt$$

$$= \frac{1}{4} \{xy - y \int_{o}^{x} \frac{\sin (m+\frac{1}{2})s}{\sin \frac{1}{2} s} ds - x \int_{o}^{x} \frac{\sin (n+\frac{1}{2})t}{\sin \frac{1}{2} t} dt$$

$$+ \int_{o,o}^{y,x} \frac{\sin (m+\frac{1}{2})s}{\sin \frac{1}{2} s} \frac{\sin (n+\frac{1}{2})t}{\sin \frac{1}{2} t} dsdt \}$$

Next we prove three lemmas which we will use in the sequel.

(1.25) Lemma 1

If $g(u,v)$ is a regular Hausdorff weight function, and $\varphi(m,n; s,t,u,v) = 0 (s^p t^q)$, $p,q > -1$, uniformly in m, n, u and v , and if φ is continuous in u and v , then

$$\int_{0,0}^{1,1} \int_{0,0}^{y,x} \varphi(m,n;\ s,t,u,v)\ dsdtd^2g(u,v) = 0\ (x^{p+1}\ y^{q+1})$$

$$= 0\ (x^{p+1}) = 0\ (y^{q+1})$$

uniformly in m,n .

Proof

Lemma 1 is obvious.

(2.16) Definition

We let

$$\rho_1 \sin \alpha = u \sin s\ , \qquad \rho_1 \cos \alpha = 1-u+u \cos s$$

$$\rho_2 \sin \beta = v \sin t\ , \qquad \rho_2 \cos \beta = 1-v+v \cos t$$

where ρ_1 , α are functions of u and s , ρ_2 , β are functions of v and t , and $0 \leq \rho_1, \rho_2 \leq 1$ for $0 \leq u, v \leq 1$, $0 \leq s, t \leq \pi$. Then

$$1-u+ue^{is} = \rho_1 e^{i\alpha}$$

$$1-v+ve^{it} = \rho_2 e^{i\beta} \quad .$$

(2.17) Lemma 2

If $g(u,v)$ is a regular Hausdorff weight function, then

$$\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1 \sin m\alpha \cot \frac{s}{2}\, ds\,dt\,d^2 g(u,v) = o\,(y)\ , \quad m = 1,2,3,\ldots\ .$$

Proof

Since under the given hypothesis,

$$\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1 \sin m\alpha \cot \frac{s}{2}\, ds\,dt\,d^2 g(u,v)$$

$$= y \int_0^1 \int_0^x \rho_1^m \sin m\alpha \cot \frac{s}{2}\, ds\,dg(u,1)\ ,$$

the problem reduces to proving that

$$\int_0^1 \int_0^x \rho_1^m \sin m\alpha \cot \frac{s}{2}\, ds\,dg(u,1) = o\,(1)\ , \quad m = 1,2,3,\ldots\ .$$

By (2.12) we have

$$h_{m,n}(\psi;x) = -\frac{1}{2}x + \frac{1}{2}\sum_{k=0}^{m}\binom{m}{k}\int_0^x \frac{\sin\,(k+\frac{1}{2})s}{\sin\frac{1}{2}\,s}\,ds \int_0^1 u^k(1-u)^{m-k}dg(u,1)$$

$$= -\frac{1}{2}x + \frac{1}{2}\int_0^1\int_0^x \sum_{k=1}^{m}\binom{m}{k}\frac{\mathrm{Im}\,e^{i(k+\frac{1}{2})s}}{\sin\frac{1}{2}\,s}\,u^k(1-u)^{m-k}ds\,dg(u,1)$$

$$= -\frac{1}{2}x + \frac{1}{2}\int_0^1\int_0^x \frac{\mathrm{Im}\,\{(1-u+ue^{is})^m\,e^{i\frac{s}{2}}\}}{\sin\frac{1}{2}\,s}\,ds\,dg(u,1)$$

$$= -\frac{1}{2}x + \frac{1}{2}\int_0^1\int_0^x \frac{\mathrm{Im}\,\rho_1 e^{i(m\alpha+\frac{s}{2})}}{\sin\frac{1}{2}\,s}\,ds\,dg(u,1)\ ,$$

where we have used the definition (2.15). Then

$$h_{m,n}(\psi;x) = -\frac{1}{2}x + \frac{1}{2}\int_0^1\int_0^x \rho_1^m \frac{\sin\,(m\alpha+\frac{s}{2})}{\sin\frac{1}{2}s}\,dsdg(u,1)$$

$$= -\frac{1}{2}x + \frac{1}{2}\int_0^1\int_0^x \rho_1^m \sin m\alpha \cot\frac{s}{2}\,dsdg(u,1)$$

$$+\frac{1}{2}\int_0^1\int_0^x \rho_1^m \cos m\alpha dsdg(u,1)$$

$$= \frac{1}{2}\int_0^1\int_0^x \rho_1^m \sin m\alpha \cot\frac{s}{2}\,dsdg(u,1) + O\,(1)\ ,$$

where the last step is justified by observing that $0 \leq \rho_1 \leq 1$. Now $h_{m,n}(\psi;x)$ is a regular transform of a bounded sequence, hence $\{h_{m,n}(\psi;x)\}$ is bounded uniformly in m,n and $0 \leq x < \pi$. Hence

$$\int_0^1\int_0^x \rho_1^m \sin m\alpha \cot\frac{s}{2}\,dsdg(u,1) = O\,(1)\ , \quad m,n = 1,2,3,\ldots\ .$$

This proves the lemma.

(2.18) Corollary 1

If $g(u,v)$ is a regular Hausdorff weight function, then

$$\sum_{k,\ell=0}^{m,n} \binom{n}{\ell}\binom{m}{k} \int_{0,0}^{y,x} \frac{\sin\,(k+\frac{1}{2})s}{\sin\frac{1}{2}s}\,ds \int_{0,0}^{1,1} u^k(1-u)^{m-k}v^\ell(1-v)^{n-\ell}d^2g(u,v)$$

$$= O\,(y)\ , \quad m,n = 1,2,3,\ldots\ .$$

(2.19) Corollary 2

Under the above hypothesis, we also have

$$\int_{0,0}^{1,1} \int_{0,0}^{x,y} \rho_1^m \frac{\sin m\alpha}{s} \, dsdtd^2 g(u,v) = 0\,(y) \;, \quad m = 1,2,3,\ldots \;.$$

Corollary 1 follows from (2.12) and the proof of Lemma 2. Corollary 2 follows from Lemma 2 by observing that

$$\left|\cot \frac{s}{2} - \frac{2}{s} \right| \leq \frac{2}{\pi} \;, \quad 0 \leq s \leq \pi \quad .$$

Then

$$\int_{0,0}^{1,1} \int_{0,0}^{y,x} \rho_1 \sin m\alpha \cot \frac{s}{2} \, dsdtd^2 g(u,v) - 2 \int_{0,0}^{1,1} \int_{0,0}^{y,x} \rho_1 \frac{\sin m\alpha}{s} \, dsdtd^2 g(u,v)$$

$$= \int_{0,0}^{1,1} \int_{0,0}^{y,x} \rho_1 \sin m\alpha \;\; \{\cot \frac{s}{2} - \frac{2}{s}\} \, dsdtd^2 g(u,v)$$

$$= 0(y)$$

by Lemma 1.

(2.20) Remarks

Lemmas 1 and 2 are also valid under the less restricted hypothesis that g(u,v) is only of bounded variation. The proof of Lemma 1 is again immediate. The proof of Lemma 2 requires a minor modification after observing that a Hausdorff summability method relative to a weight function

$g(u,v)$ takes bounded sequences into bounded sequences whenever $g(u,v)$ is of bounded variation. The next lemma, however, is easier to prove under the less restricted hypothesis.

(2.21) Lemma 3

If $g(u,v)$ is a given function of bounded variation, and if $\varphi(n;t,v) = O(t^p)$, $p > -1$, is continuous in v , and uniformly bounded in n,v, then

$$\int_{0,0}^{1,1} \int_{0,0}^{y,x} \rho_1^m \frac{\sin m\alpha}{s} \varphi(n;t,v)\, ds dt d^2 g(u,v) = O\,(y^{p+1}) \quad ,$$

$$m,n = 1,2,3,\ldots \quad .$$

Proof

Since $g(u,v)$ is of bounded variation, it can be expressed as the difference of two positively monotonic functions. Hence without loss of generality, we assume that it is already a positively monotonic function. Also, since otherwise we could split $\varphi(n;t,v)$ into the negative and positive parts and treat each part separately, we assume that $\varphi(n;t,v)$ is non-negative. We write

$$(2.22) \qquad d^2 f(n;y,u,v) = \int_0^y \varphi(n;t,v) dt \cdot d^2 g(u,v) \quad ,$$

and observe that $\varphi(n;t,v) = O\,(t^p)$, so that $0 \leq \varphi(n;t,v) \leq M' t^p$ for some constant M' . Then

$$V(f_n) \le My^{p+1} \, V(g) \;, \quad f_n = f(n;y,u,v) \tag{2.23}$$

and

$$d^2 f(n;y,u,v) \le My^{p+1} d^2 g(u,v) \quad , \tag{2.24}$$

where $(p+1)M = M'$. Thus for each fixed y , there exists a function $f(n;y,u,v)$, for which (2.22) is satisfied, and the variation of the sequence of functions $\{f(n;y,u,v)\}$ is bounded uniformly in n , and in fact uniformly for $0 < y < \pi$, by (2.23) . Hence the sequence of Hausdorff transforms, relative to the sequence of weight functions $\{f(n;y,u,v)\}$, will take bounded sequences into bounded sequences.

Let $\{s_{k\ell}(x)\}$ be any non-negative, bounded sequence which is independent of y . Again, we assume the non-negative property as a matter of convenience only. Relative to the weight function $g(u,v)$, the mn-th Hausdorff transform of this sequence is given by

$$h_{m,n}(g;x) = \sum_{k,\ell=o}^{m,n} \binom{n}{\ell}\binom{m}{k} s_{k,\ell}(x) \int_{o,o}^{1,1} u^k(1-u)^{m-k} \, v^\ell(1-v)^{n-\ell} \, d^2 g(u,v) \; .$$

Now relative to a function $f(i;y,u,v)$ of the sequence $\{f(n;y,u,v)\}$, the mn-th Hausdorff transform of the sequence $\{s_{k,\ell}(x)\}$ is given by

$$h_{m,n}(f_i;x) = \sum_{k,\ell=o}^{m,n} \binom{n}{\ell}\binom{m}{k} s_{k,\ell}(x) \int_{o,o}^{1,1} u^k(1-u)^{m-k} \, v^\ell(1-v)^{n-\ell} d^2 f(i;y,u,v)$$

$$\le My^{p+1} \sum_{k,\ell=o}^{m,n} \binom{n}{\ell}\binom{m}{k} s_{k,\ell}(x) \int_{o,o}^{1,1} u^k(1-u)v^\ell(1-v)^{n-\ell} d^2 g(u,v)$$

by (2.24), since all the terms on the right are non-negiative. Thus

$$h_{m,n}(f_i;x) \leq My^{p+1}\ h_{m,n}(g;x) \quad .$$

Now the sequence $\{s_{k,\ell}(x)\}$ is bounded, and so also the sequence $\{h_{m,n}(g;x)\}$ is bounded. Then

(2.25) $$h_{m,n}(f_i;x) = 0\ (y^{p+1}) \quad .$$

To complete the proof of the lemma, we let

$$s_{k\ell}(x) = \sum_{j=1}^{k} \frac{\sin jx}{j}$$

$$= -\frac{1}{2}x + \frac{1}{2}\int_{o}^{x} \frac{\sin (k+\frac{1}{2})s}{\sin \frac{1}{2}s}\ ds \quad ,$$

so that

$$h_{m,n}(f_n;x) = \sum_{k,\ell=o}^{m,n} \binom{n}{\ell}\binom{m}{k} \{-\frac{1}{2}x + \frac{1}{2}\int_{o}^{\pi} \frac{\sin (k+\frac{1}{2})s}{\sin \frac{1}{2}s}\ ds\} \cdot$$

$$\cdot \int_{o,o}^{1,1} u^k(1-u)^{m-k}v^{\ell}(1-v)^{n-\ell}d^2f(n;y,u,v)$$

$$= -\frac{1}{2}x \int_{o,o}^{1,1} d^2f(n;y,u,v) + \sum_{k=o}^{m}\binom{m}{k}\frac{1}{2}\int_{o}^{x}\frac{\operatorname{Im} e^{i(k+\frac{1}{2})s}}{\sin\frac{1}{2}s} \cdot$$

$$\cdot \int_{o,o}^{1,1} u^k(1-u)^{m-k}d^2f(n;y,u,v)$$

$$= \frac{1}{2}\int_{0,0}^{1,1}\int_{0}^{x} \frac{\mathrm{Im}\{(1-u+ue^{is})^{m}\, e^{i\frac{s}{2}}\}}{\sin\frac{1}{2}\,s}\, dsd^{2}f(n;y,u,v) + 0\ (y^{p+1})$$

by (2.23) ;

$$= \frac{1}{2}\int_{0,0}^{1,1}\int_{0}^{x} \frac{\mathrm{Im}\ \rho_{1}^{m}\, e^{i(m\alpha+\frac{s}{2})}}{\sin\frac{1}{2}}\ dsd^{2}f(n;y,u,v) + 0\ (y^{p+1})$$

$$= \frac{1}{2}\int_{0,0}^{1,1}\int_{0}^{x} \rho_{1}^{m}\,\frac{\sin\,(m\alpha+\frac{s}{2})}{\sin\frac{s}{2}}\ dsd^{2}f(n;y,u,v) + 0\ (y^{p+1})$$

$$= \frac{1}{2}\int_{0,0}^{1,1}\int_{0}^{x} \rho_{1}^{m}\ \sin m\alpha\ \cot\frac{s}{2}\ dsd^{2}f(n;y,u,v)$$

$$+ \frac{1}{2}\int_{0,0}^{1,1}\int_{0}^{x} \rho_{1}^{m}\ \cos m\alpha dsd^{2}f(n;y,u,v) + 0\ (y^{p+1})$$

$$= \frac{1}{2}\int_{0,0}^{1,1}\int_{0}^{x} \rho_{1}^{m}\ \sin m\alpha\ \{\cot\frac{s}{2} - \frac{s}{2} + \frac{2}{s}\}\ dsd^{2}f(n;y,u,v) + 0\ (y^{p+1})$$

$$= \int_{0,0}^{1,1}\int_{0}^{x} \rho_{1}^{m}\,\frac{\sin m\alpha}{s}\int_{0}^{y} \varphi(n;t,v)dsdtd^{2}g(u,v) + 0\ (y^{p+1})$$

$$= \int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_{1}^{m}\,\frac{\sin m\alpha}{s}\ \varphi(n;t,v)\ dsdtd^{2}g(u,v) + 0\ (y^{p+1})$$

$$= 0\ (y^{p+1})$$

by (2.25). This completes the proof of the lemma.

(2.26) Corollary

If $g(u,v)$ is of bounded variation and if $\varphi(n;t,v)$ is a given function which is continuous in v and uniformly bounded in n,t,v, then

$$\int_{o,o}^{1,1}\int_{o,o}^{y,x} \rho_1^m \frac{\sin m\alpha}{s} \varphi(n;t,v)\,ds\,dt\,d^2g(u,v) = 0\,(y)\ , \quad m,n = 1,2,3,\ldots\ .$$

(2.27) Remarks

In the applications that we have in mind, the function $\varphi(n;t,v)$ is continuous in t and v. However, for the purposes of the lemma, continuity in t is not required, and continuity in v may be relaxed to requiring continuity only at the points of discontinuity of the function $g(u,v)$.

Lemma 3 is clearly valid if we replace α, m, n, ρ_1, s, t, u, v, x and y by β, n, m, ρ_2, t, s, v, u, y and x respectively.

§ 2.

We are now in a position to extend the theorems (2.5) and (2.6) to the two dimensional case for the function $\varphi(x,y) = \psi(x)\,\psi(y)$. We consider the regular Hausdorff means of the partial sums $s_{m,n}(\varphi;x,y)$, given by (2.14), relative to the regular Hausdorff weight function $g(u,v)$. Denoting these means, as before, by $h_{m,n}(\varphi;x,y)$, we have by (2.7) and (2.14),

$$h_{m,n}(\varphi;x,y) = \frac{1}{4}\sum_{k,\ell=0}^{m,n}\binom{n}{\ell}\binom{m}{k}\{xy-y\int_0^x \frac{\sin(k+\frac{1}{2})s}{\sin\frac{1}{2}s}\,ds - x\int_0^y \frac{\sin(\ell+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$+\int_{0,0}^{y,x}\frac{\sin(k+\frac{1}{2})s}{\sin\frac{1}{2}s}\frac{\sin(\ell+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dsdt\}\int_{0,0}^{1,1}u^k(1-v)^{m-k}v^\ell(1-v)^{n-\ell}d^2g(u,v)$$

$$= \frac{1}{4}xy - \frac{1}{4}\sum_{k,\ell=0}^{m,n}\binom{n}{\ell}\binom{m}{k}\int_{0,0}^{y,x}\frac{\sin(k+\frac{1}{2})s}{\sin\frac{1}{2}s}\,dsdt\int_{0,0}^{1,1}u^k(1-u)^{m-k}v^\ell(1-v)^{n-\ell}d^2g(u-v)$$

$$-\frac{1}{4}\sum_{k,\ell=0}^{m,n}\binom{n}{\ell}\binom{m}{k}\int_{0,0}^{y,x}\frac{\sin(\ell+\frac{1}{2})s}{\sin\frac{1}{2}s}\,dsdt\int_{0,0}^{1,1}u^k(1-u)^{m-k}v^\ell(1-v)^{n-\ell}d^2g(u,v)$$

$$+\frac{1}{4}\sum_{k,\ell=0}^{m,n}\binom{n}{\ell}\binom{m}{k}\int_{0,0}^{y,x}\frac{\sin(k+\frac{1}{2})s}{\sin\frac{1}{2}s}\frac{\sin(\ell+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dsdt\int_{0,0}^{1,1}u^k(1-u)^{m-k}v^\ell(1-v)^{n-\ell}d^2g(u,v)$$

$$=\frac{1}{4}\sum_{k,\ell=0}^{m,n}\binom{n}{\ell}\binom{m}{k}\int_{0,0}^{y,x}\frac{\sin(k+\frac{1}{2})s}{\sin\frac{1}{2}s}\frac{\sin(\ell+\frac{1}{2})t}{\sin\frac{1}{2}s}\,dsdt\int_{0,0}^{1,1}u^k(1-u)^{m-k}v^\ell(1-v)^{n-\ell}d^2g(u,v)$$

$$+ O(x) + O(y) \quad ,$$

where we have used Corollary 1 of Lemma 2. With ρ_1 , ρ_2 , α and β defined in (2.16),

$$h_{m,n}(\varphi;x,y) = \frac{1}{4}\sum_{k,\ell=0}^{m,n}\binom{n}{\ell}\binom{m}{k}\int_{0,0}^{y,x}\frac{\operatorname{Im} e^{i(k+\frac{1}{2})s}}{\sin\frac{1}{2}s}\frac{\operatorname{Im} e^{i(\ell+\frac{1}{2})t}}{\sin\frac{1}{2}t}\,dsdt\,\cdot$$

$$\cdot\int_{0,0}^{1,1}u^k(1-u)^{m-k}v^\ell(1-v)^{n-\ell}d^2g(u,v)$$

$$+ O(x) + O(y)$$

$$= \frac{1}{4}\int_{0,0}^{1,1}\int_{0,0}^{y,x} \frac{\operatorname{Im}\{(1-u+ue^{is})^{m}\, e^{i\frac{s}{2}}\}}{\sin\frac{1}{2}s}\, \frac{\operatorname{Im}\{(1-v+ve^{it})^{n}\, e^{i\frac{t}{2}}\}}{\sin\frac{1}{2}t}\, ds\,dt\,d^{2}g(u,v)$$

$$+ O(x) + O(y)$$

$$= \frac{1}{4}\int_{0,0}^{1,1}\int_{0,0}^{y,x} \frac{\operatorname{Im} \rho_1^{m}\, e^{i(m\alpha+\frac{s}{2})}}{\sin\frac{1}{2}s}\, \frac{\operatorname{Im} \rho_2^{n}\, e^{i(n\beta+\frac{t}{2})}}{\sin\frac{1}{2}t}\, ds\,dt\,d^{2}g(u,v)$$

$$+ O(x) + O(y)$$

$$= \frac{1}{4}\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^{m}\,\rho_2^{n}\, \frac{\sin(m\alpha+\frac{s}{2})}{\sin\frac{1}{2}s}\, \frac{\sin(n\beta+\frac{t}{2})}{\sin\frac{1}{2}t}\, ds\,dt\,d^{2}g(u,v)$$

$$+ O(x) + O(y)$$

$$= \frac{1}{4}\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^{m}\rho_2^{n}\, \{\sin m\alpha \cot\frac{s}{2} + \cos m\alpha\}\{\sin n\beta \cot\frac{t}{2} + \cos n\beta\}\,\cdot$$

$$\cdot\, ds\,dt\,d^{2}g(u,v)$$

$$+ O(x) + O(y)$$

$$= \frac{1}{4}\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^{m}\rho_2^{n} \sin m\alpha \sin n\beta \cot\frac{s}{2} \cot\frac{t}{2}\, ds\,dt\,d^{2}g(u,v)$$

$$+ \frac{1}{4}\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^{m}\rho_2^{n} \sin m\alpha \cos n\beta \cot\frac{s}{2}\, ds\,dt\,d^{2}g(u,v)$$

$$+ \frac{1}{4}\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m\rho_2^n \cos m\alpha \sin n\beta \cot \frac{t}{2}\, dsdtd^2g(u,v)$$

$$+ \frac{1}{4}\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m\rho_2^n \cos m\alpha \cos n\beta\, dsdtd^2g(u,v) + O(x) + O(y)$$

$$= \frac{1}{4}\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m\rho_2^n \sin m\alpha \sin n\beta \left\{\cot \frac{s}{2} - \frac{2}{s} + \frac{2}{s}\right\}\left\{\cot \frac{t}{2} - \frac{2}{t} + \frac{2}{t}\right\} \cdot$$

$$\cdot\; dsdtd^2g(u,v) + O(x) + O(y) \quad ,$$

where we applied Lemma 1 to the last integral, and observed that

$$\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m\rho_2^n \sin m\alpha \cos n\beta \cot \frac{s}{2}\, dsdtd^2g(u,v)$$

$$= \int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m\rho_2^n \sin m\alpha \cos n\beta \left\{\cot \frac{s}{2} - \frac{2}{s} + \frac{2}{s}\right\} dsdtd^2g(u,v)$$

$$= 2\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m \frac{\sin m\alpha}{s} \rho_2^n \cos n\beta\, dsdtd^2g(u,v)$$

$$+ \int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m\rho_2^n \sin m\alpha \cos n\beta \left\{\cot \frac{s}{2} - \frac{2}{s}\right\} dsdt\, d^2g(u,v)$$

$$= O(y)$$

by Lemma 3 with $p = 0$, and Lemma 1. Then

$$(2.27)\qquad h_{m,n}(\varphi;x,y) = \int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m\rho_2^n \frac{\sin m\alpha}{s}\,\frac{\sin n\beta}{t}\, dsdtd^2g(u,v)$$

$$+ \frac{1}{2}\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m\rho_2^n \frac{\sin m\alpha}{s} \sin n\beta \left\{\cot\frac{t}{2} - \frac{2}{t}\right\} dsdtd^2g(u,v)$$

$$+ \frac{1}{2}\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m\rho_2^n \sin m\alpha \frac{\sin n\beta}{t}\left\{\cot\frac{s}{2} - \frac{2}{s}\right\} dsdtd^2g(u,v)$$

$$+ \frac{1}{4}\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m\rho_2^n \sin m\alpha \sin n\beta \left\{\cot\frac{s}{2} - \frac{2}{s}\right\}\left\{\cot\frac{t}{2} - \frac{2}{t}\right\} \cdot$$

$$\cdot\, dsdtd^2g(u,v)$$

$$+ 0\,(x) + 0\,(y)$$

$$= \int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m\rho_2^n \frac{\sin m\alpha}{s}\,\frac{\sin n\beta}{t}\, dsdtd^2g(u,v) + 0\,(x) + 0\,(y) \quad ,$$

again by Lemma 3 with $p = 0$, and Lemma 1. Now for $i = 1,2$,

$$1-\rho_i^k = (1-\rho_i)\sum_{j=1}^{k-1} \rho_i^j \le k(1-\rho_i) \le k(1-\rho_i^2) \ ,$$

and so

$$1-\rho_i^k = \lambda_i k(1-\rho_i^2) \ , \quad 0 < \lambda_i < 1 \quad ,$$

so that

$$\rho_1^m = 1 - \lambda_1 m(1-\rho_1^2)$$

$$\rho_2^n = 1 - \lambda_2 n(1-\rho_2^2),$$

and using (2.16),

$$1 - \rho_1^2 = 4u(1-u)\sin^2\frac{s}{2} \le s^2 \tag{2.28}$$

$$1 - \rho_2^2 = 4v(1-v)\sin^2\frac{t}{2} \le t^2 \quad .$$

Then

$$h_{m,n}(\varphi;x,y) = \int_{o,o}^{1,1}\int_{o,o}^{y,x} \{1-\lambda_1 m(1-\rho_2^2)\}\,\{1-\lambda_2 n(1-\rho_2^2)\}\,\cdot$$

$$\cdot\ \frac{\sin m\alpha}{s}\,\frac{\sin n\beta}{t}\,dsdtd^2g(u,v) + O(x) + O(y)$$

$$= \int_{o,o}^{1,1}\int_{o,o}^{y,x} \frac{\sin m\alpha}{s}\,\frac{\sin n\beta}{t}\,dsdtd^2g(u,v)$$

$$- m\int_{o,o}^{1,1}\int_{o,o}^{y,x} \lambda_1(1-\rho_1^2)\,\frac{\sin m\alpha}{s}\,\frac{\sin n\beta}{t}\,dsdtd^2g(u,v)$$

$$- n\int_{o,o}^{1,1}\int_{o,o}^{y,x} \lambda_2(1-\rho_2^2)\,\frac{\sin m\alpha}{s}\,\frac{\sin n\beta}{t}\,dsdtd^2g(u,v)$$

$$+ mn\int_{0,0}^{1,1}\int_{0,0}^{y,x} \lambda_1\lambda_2\ (1-\rho_1^2)(1-\rho_2^2)\ \frac{\sin m\alpha}{s}\,\frac{\sin n\beta}{t}\ dsdtd^2g(u,v)$$

$$+ O(x) + O(y)\quad .$$

Now by (2.28),

$$\lambda_1\lambda_2(1-\rho_1^2)(1-\rho_2^2)\ \frac{\sin m\alpha}{s}\,\frac{\sin n\beta}{t} = O(st)$$

$$\lambda_1(1-\rho_1^2)\ \frac{\sin m\alpha}{s} = O(s)$$

$$\lambda_2(1-\rho_2^2)\ \frac{\sin n\beta}{t} = O(t)\quad ,$$

so that, by Lemmas 1 and 3,

$$h_{m,n}(\varphi;x,y) = \int_{0,0}^{1,1}\int_{0,0}^{y,x} \frac{\sin m\alpha}{s}\ \frac{\sin n\beta}{t}\ dsdtd^2g(u,v) + O(mx^2)$$

$$+ O(my^2) + O(mnx^2y^2) + O(x) + O(y)\quad ,$$

and if $mx_m \to \tau_1 < \infty$, and $ny_n \to \tau_2 < \infty$ as $m,n \to \infty$ and $x_m, y_n \to 0$, then

$$(2.29)\qquad h_{m,n}(\varphi;x_m,y_n) = \int_{0,0}^{1,1}\int_{0,0}^{y_n,x_m} \frac{\sin m\alpha}{s}\ \frac{\sin n\beta}{t}\ dsdtd^2g(u,v)$$

$$+ O(x_m) + O(y_n)$$

Now [10, p. 445]

$$\alpha = \frac{\arctan u \sin s}{1-u(1-\cos s)} = \arctan z$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^{2k-1}}{2k-1} = us + O(us^3) \quad .$$

Then, since

$$\sin a - \sin b = 2 \cos \frac{1}{2} (a+b) \sin \frac{1}{2} (a-b),$$

it follows that

$$\sin m\alpha - \sin mus = 2 \cos \frac{m}{2} (\alpha+us) \sin O(mus^3)$$

$$= O(mus^3) \cos \frac{m}{2} (\alpha+us)$$

$$= O(mus^3)$$

and

$$\sin m\alpha = \sin mus + O(mus^3) \quad .$$

Similarly,

$$\sin n\beta = \sin ntv + O(nvt^3) \; .$$

Substituting in (2.29), and dropping the subcripts for a moment,

$$h_{m,n}(\varphi;x,y) = \int_{o,o}^{1,1} \int_{o,o}^{y,x} \{\sin mus + O(mus^3)\}\{\sin ntv + O(nvt^3)\} \cdot$$

$$\cdot \frac{ds}{s} \frac{dt}{t} d^2g(u,v) + O(x) + O(y)$$

$$= \int_{0,0}^{1,1} \int_{0,0}^{y,x} \frac{\sin msu}{s} \frac{\sin ntv}{t} \, dsdtd^2g(u,v)$$

$$+ \int_{0,0}^{1,1} \int_{0,0}^{y,x} O(mus^2) \frac{\sin ntv}{t} \, dsdtd^2g(u,v)$$

$$+ \int_{0,0}^{1,1} \int_{0,0}^{y,x} O(nvt^2) \frac{\sin msu}{s} \, dsdtd^2g(u,v)$$

$$+ \int_{0,0}^{1,1} \int_{0,0}^{y,x} O(mnuvs^2t^2) \, dsdtd^2g(u,v) + O(x) + O(y)$$

$$= \int_{0,0}^{1,1} \int_{0,0}^{y,x} \frac{\sin msu}{s} \frac{\sin ntv}{t} \, dsdtd^2g(u,v) + O(x) + O(y)$$

$$= \int_{0,0}^{1,1} \int_{0,0}^{ny,mx} \frac{\sin su}{s} \frac{\sin tv}{t} \, dsdtd^2g(u,v) + O(x) + O(y) ,$$

$$mx \to \tau_1 < \infty , \qquad ny \to \tau_2 < \infty ,$$

by Lemmas 1 and 3, since under the assumed conditions, $O(mnuvs^2t^2) = O(st)$, $O(mus^2) = O(s)$ and $O(nvt^2) = O(t)$. We summarize the result in a theorem.

(2.30) Theorem 2

If $mx \to \tau_1 < \infty$, $ny \to \tau_2 < \infty$ as $m,n \to \infty$, $x,y \to 0$,

then

$$h_{m,n}(\varphi;x,y) \to \int_{0,0}^{1,1} \int_{0,0}^{\tau_2,\tau_1} \frac{\sin su}{s} \frac{\sin tv}{t} \, dsdtd^2g(u,v) \quad .$$

§ 3.

Next, we consider the case where $mx \to \infty$, $ny \to \infty$ as $m,n \to \infty$ and $x,y \to 0$. To this end we use a result of Szász [10, p. 446].

(2.31) Lemma

$$\rho_1^m = e^{-\frac{m}{2}(1-\rho_1^2)} - m\lambda_1(1-\rho_1^2)^2 \, e^{-\frac{m}{2}(1-\rho_1^2)}$$

$$= e^{-2mu(1-u)\sin^2 \frac{s}{2}} - m\lambda_1(1-\rho_1^2)^2 \, e^{-\frac{m}{2}(1-\rho_1^2)} \quad , \quad 0 < \lambda_1 < 1 \ .$$

Now observe that

$$0 \leq (\tfrac{s}{2})^2 - \sin^2 \tfrac{s}{2} = (\tfrac{s}{2})^2 - \tfrac{1}{2}\,(1-\cos s)$$

$$= \tfrac{1}{2}\,\{\tfrac{s^4}{4!} - \tfrac{s^6}{6!} + \ldots\}$$

$$\leq \tfrac{1}{2}\,(\tfrac{s}{2})^4 \quad .$$

Then if $a \geq 0$,

$$e^{a(\frac{s}{2})^2 - a\sin^2\frac{s}{2}} - 1 \leq e^{\frac{a}{2}(\frac{s}{2})^4} - 1$$

$$= e^{\frac{a}{2}\left(\frac{s}{2}\right)^4}\left\{1-e^{-\frac{a}{2}\left(\frac{s}{2}\right)^4}\right\}$$

$$< \frac{a}{2}\left(\frac{s}{2}\right)^4 e^{\frac{a}{2}\left(\frac{s}{2}\right)^4}$$

$$= O(as^4) \quad , \qquad 0 \le s < \pi \quad .$$

Then

$$e^{a\left(\frac{s}{2}\right)^2 - a\sin^2\frac{s}{2}} - 1 = O(as^4) \quad ,$$

and

$$e^{-a\sin^2\frac{s}{2}} = e^{-a\left(\frac{s}{2}\right)^2} + O(as^4)$$

With

$$a = 2mu(1-u) \ ,$$

(2.32)
$$e^{-2mu(1-u)\sin^2\frac{s}{2}} = e^{-\frac{m}{2}u(1-u)s^2} + O(ms^4) \quad .$$

Next,

$$\left(\frac{s}{\pi}\right)^2 \le \sin^2\frac{s}{2} \le \left(\frac{s}{2}\right)^2 \quad ,$$

so that

$$\left(\frac{2s}{\pi}\right)^2 u(1-u) \le 1-\rho_1^2 \le s^2u(1-u)$$

and

$$m\lambda_1(1-\rho_1^2)^2\, e^{-\frac{m}{2}(1-\rho_1^2)} \leq m\lambda_1 s^4 u^2(1-u)^2\, e^{-\frac{2ms^2}{\pi^2}u(1-u)}$$

Using this result, together with (2.31) and (2.32),

$$\rho_1^m = e^{-\frac{m}{2}u(1-u)s^2} + O(ms^4) \quad . \tag{2.33}$$

Then

$$\rho_1^m\rho_2^n \frac{\sin m\alpha}{s}\frac{\sin n\beta}{t} = e^{-\frac{m}{2}u(1-u)s^2}\, e^{-\frac{n}{2}v(1-v)t^2}\frac{\sin m\alpha}{s}\frac{\sin n\beta}{t}$$

$$+ O(ms^3)\,\rho_2^n\frac{\sin n\beta}{t} + O(nt^3)\,\rho_1^m\frac{\sin m\alpha}{s} + O(mns^3t^3)$$

$$= \psi(m,n,st,u,v) + O(ms^3)\,\rho_2^n\frac{\sin n\beta}{t}$$

$$+ O(nt^3)\,\rho_1^m\frac{\sin m\alpha}{s} + O(mns^3t^3) \ .$$

Then by (2.27),

$$h_{m,n}(\varphi;x,y) = \int_{0,0}^{1,1}\int_{0,0}^{y,x}\psi(m,n,s,t,u,v)\; dsdtd^2g(u,v)$$

$$+ \int_{0,0}^{1,1}\int_{0,0}^{y,x} O(ms^3)\,\rho_2^n\frac{\sin n\beta}{t}\; dsdtd^2g(u,v)$$

$$+ \int_{0,0}^{1,1}\int_{0,0}^{y,x} O(nt^3)\,\rho_1^m\frac{\sin m\alpha}{s}\; dsdtd^2g(u,v)$$

$$+ \int_{0,0}^{1,1} \int_{0,0}^{y,x} O(mns^3t^3)\, dsdtd^2g(u,v) + O(x) + O(y)$$

$$= \int_{0,0}^{1,1} \int_{0,0}^{y,x} \psi(m,n,s,t,u,v)\, dsdtd^2g(u,v) + O(mx^4) + O(ny^4)$$

$$+ O(mnx^4y^4) + O(x) + O(y) \quad ,$$

where again we have applied Lemmas 1 and 3 . Now restricting the argument to the case where, as $m,n \to \infty$ and $x,y \to 0$,

$$mx^2 = O(1) \ , \quad ny^2 = O(1) \quad , \tag{2.34}$$

we have

$$h_{m,n}(\varphi;x,y) = \int_{0,0}^{1,1} \int_{0,0}^{y,x} \psi(m,n,s,t,u,v)\, dsdtd^2g(u,v) \tag{2.35}$$

$$+ O(x) + O(y) \quad .$$

We remark that this restriction still permits that $mx \to \infty$, $ny \to \infty$ as $m,n \to \infty$ and $x,y \to 0$.

Next, we use a lemma due to Szász [10, p. 448] to reduce (2.35) further.

(2.36) Lemma

For small values of s and for $0 \leq u \leq 1$,

$$\alpha = \alpha(s,u) = su + O(u(1-u)s^3)$$
$$= su + O(s^3) \quad .$$

Then again using the identity

$$\sin a - \sin b = 2 \cos \frac{1}{2}(a+b) \sin \frac{1}{2}(a-b) \quad ,$$

we have

$$\sin m\alpha - \sin msu = 2 \cos (msu + O(ms^3)) \sin O(ms^3)$$
$$= O(ms^3) \quad ,$$

or

$$\sin m\alpha = \sin msu + O(ms^3) \quad .$$

Then

$$\sin m\alpha \sin n\beta = \sin msu \sin ntv + O(nt^3) \sin m\alpha$$
$$+ O(ms^3) \sin n\beta + O(mns^3t^3)$$

and

$$(2.37) \quad \int_{0,0}^{1,1} \int_{0,0}^{y,x} \psi(m,n,s,t,u,v) \, dsdtd^2g(u,v)$$
$$= \int_{0,0}^{1,1} \int_{0,0}^{y,x} e^{-\frac{m}{2} u(1-u)s^2} e^{-\frac{n}{2} v(1-v)t^2} \frac{\sin m\alpha}{s} \frac{\sin n\beta}{t} \, dsdtd^2g(u,v)$$

$$= \int_{0,0}^{1,1} \int_{0,0}^{y,x} e^{-\frac{m}{2} u(1-u)s^2} e^{-\frac{n}{2} v(1-v)t^2} \frac{\sin msu}{s} \frac{\sin ntv}{t} \, dsdtd^2g(u,v)$$

$$+ \int_{0,0}^{1,1} \int_{0,0}^{y,x} O(nt^2) \, e^{-\frac{m}{2} u(1-u)s^2} \frac{\sin m\alpha}{s} \, dsdtd^2g(u,v)$$

$$+ \int_{0,0}^{1,1} \int_{0,0}^{y,x} O(ms^2) \, e^{-\frac{n}{2} v(1-v)t^2} \frac{\sin n\beta}{t} \, dsdtd^2g(u,v)$$

$$+ \int_{0,0}^{1,1} \int_{0,0}^{y,x} O(mns^2t^2) \, dsdtd^2g(u,v) \quad .$$

Now by our hypothesis (2.34), $O(mns^2t^2) = O(1)$, so that by Lemma 1, the last integralis $O(xy) = O(x)$. Also, by (2.33),

$$e^{-\frac{m}{2} u(1-u)s^2} \frac{\sin m\alpha}{s} = \rho_1^m \frac{\sin m\alpha}{s} + O(ms^3) \quad .$$

Then

$$\int_{0,0}^{1,1} \int_{0,0}^{y,x} O(nt^2) \, e^{-\frac{m}{2} u(1-u)s^2} \frac{\sin m\alpha}{s} \, dsdtd^2g(u,v)$$

$$= \int_{0,0}^{1,1} \int_{0,0}^{y,x} \rho_1^m \frac{\sin m\alpha}{s} \; O(nt^2) \, dsdtd^2g(u,v)$$

$$+ \int_{0,0}^{1,1} \int_{0,0}^{y,x} O(mns^3t^2) \, dsdtd^2g(u,v)$$

$$= O(ny^3) + O(mnx^4y^3)$$

$$= O(x) + O(y)$$

by Lemmas 1 and 3, and the condition (2.33). Then also,

$$\int_{0,0}^{1,1} \int_{0,0}^{y,x} O(ms^2)\, e^{-\frac{n}{2} v(1-v)t^2} \frac{\sin n\beta}{t}\, dsdtd^2g(u,v)$$

$$= O(x) + O(y) \quad .$$

Then by (2.35) and (2.37),

$$h_{m,n}(\varphi;x,y) = \int_{0,0}^{1,1} \int_{0,0}^{y,x} e^{-\frac{m}{2} u(1-u)s^2}\, e^{-\frac{n}{2} v(1-v)t^2} \frac{\sin msu}{s} \frac{\sin ntv}{t} \cdot$$

$$\cdot\, dsdtd^2g(u,v)$$

$$= O(x) + O(y) \quad .$$

Next, we introduce another result due to Szász [10, p. 451].

(2.38) Lemma

The integral

$$\int_0^x e^{-\frac{m}{2} u(1-u)s^2} \frac{\sin msu}{s}\, ds$$

converges to $\frac{\pi}{2}$ uniformly in $0 < \epsilon \leq u \leq 1$, and boundedly in $0 \leq u \leq 1$,

as $m \to \infty$.

Then

$$\left| \int_{0,0}^{1,1} \int_{0,0}^{y,x} e^{-\frac{m}{2} u(1-u)s^2} e^{-\frac{n}{2} v(1-v)t^2} \frac{\sin msu}{s} \frac{\sin ntv}{t} \cdot \right. \tag{2.39}$$

$$\left. td^2g(u,v) \right| \leq M < \infty \quad ,$$

for $m,n = 1,2,3,\ldots$, and $0 \leq u,v \leq 1$. Then for $0 < \epsilon_1,\epsilon_2 < 1$,

$$h_{m,n}(\varphi;x,y) = \left\{ \int_{0,0}^{\epsilon_2,\epsilon_1} + \int_{\epsilon_2,0}^{1,\epsilon_1} + \int_{0,\epsilon_1}^{\epsilon_2,1} + \int_{\epsilon_2,\epsilon_1}^{1,1} \right\} \cdot$$

$$\cdot \int_{0,0}^{y,x} e^{-\frac{m}{2} u(1-u)s^2} e^{-\frac{n}{2} v(1-v)t^2} \frac{\sin msu}{s} \frac{\sin ntv}{t} \, dsdtd^2g(u,v)$$

$$+ O(x) + O(y) \ .$$

Now by (2.39), the first three integrals above are, respectively,

$$O\left(\int_{0,0}^{\epsilon_2\epsilon_1} |d^2g(u,v)| \right) , \qquad O\left(\int_{\epsilon_2,0}^{1,\epsilon_1} |d^2g(u,v)| \right)$$

and $\quad O\left(\int_{0,\epsilon_1}^{\epsilon_2,1} |d^2g(u,v)| \right)$,

and so

$$h_{m,n}(\varphi;x,y) \to \int_{\epsilon_2,\epsilon_1}^{1,1} \frac{\pi^2}{4}\, d^2g(u,v) + O\left(\int_{o,o}^{\epsilon_2,\epsilon_1} |d^2g(u,v)| \right)$$

$$+ O\left(\int_{\epsilon_2,o}^{1,\epsilon_1} |d^2g(u,v)| \right) + O\left(\int_{o,\epsilon_1}^{\epsilon_2,1} |d^2g(u,v)| \right) ,$$

$$m,n \to \infty \ ,$$

since $x,y \to 0$ as $m,n \to \infty$. Since ϵ_1 and ϵ_2 can be taken arbitrarily small,

$$h_{m,n}(\varphi;x,y) = \frac{\pi^2}{4} \int_{o^+,o^+}^{1,1} d^2g(u,v) + O\left(\int_{o,o}^{o^+,o^+} |d^2g(u,v)| \right)$$

$$+ O\left(\int_{o^+,o}^{1,o^+} |d^2g(u,v)| \right) + O\left(\int_{o,o^+}^{o^+,1} |d^2g(u,v)| \right)$$

$$\to \frac{\pi^2}{4} \ , \quad m,n \to \infty$$

since

$$\int_{o^+,o^+}^{1,1} d^2g(u,v) = 1 \quad \text{and} \quad g(o,v) = g(o^+,v) = g(u,o) = g(u,o^+).$$

We state the result in a theorem.

(2.40) Theorem 3

If $mx \to \infty$, $ny \to \infty$, $mx^2 = O(1)$, $ny^2 = O(1)$, as $m,n \to \infty$

and $x,y \to 0$, then

$$h_{m,n}(\varphi;x,y) \to \frac{\pi^2}{4} , \qquad m,n \to \infty .$$

Combining Theorems 2 and 3 , we get

(2.41) Theorem 4

If $mx \to \tau_1 \leq \infty$, $ny \to \tau_2 \leq \infty$, $mx^2 = O(1)$, $ny^2 = O(1)$, as $m,n \to \infty$ and $x,y \to 0$, then

$$h_{m,n}(\varphi;x,y) \to \int_{0,0}^{1,1} \int_{0,0}^{\tau_2,\tau_1} \frac{\sin su}{s} \frac{\sin tv}{t} \, ds dt d^2 g(u,v) , \quad m,n \to \infty .$$

(2.42) Remarks

Strictly speaking, before stating Theorem 4, we should consider the case $mx \to \tau_1 < \infty$, $ny \to \infty$. It is clear, however, that the argument would involve both methods of proof in a predictable manner, and it is equally obvious what the result would be.

(2.43) Theorem 5

If $h_{m,n}(\varphi;x,y)$ denotes the mn-th Hausdorff transform of the sequence of partial sums of the function $\varphi(x,y) = \psi(x)\,\psi(y)$, where $\psi(x)$ is defined by (2.4), then

$$\limsup_{\substack{m,n \to \infty \\ x,y \to o}} h_{m,n}(\varphi;x,y) = \max_{\tau_1,\tau_2 > o} \int_{o,o}^{1,1} \frac{\sin \tau_1 u}{u} \frac{\sin \tau_2 v}{v} g(1,1;u,v)dudv .$$

If this maximum is attained for $\tau_1 = \tau_1'$, $\tau_2 = \tau_2'$, then

$$\limsup_{\substack{m,n \to \infty \\ x,y \to o}} h_{m,n}(\varphi;x,y) = \lim_{\substack{mx_m \to \tau_1' \\ ny_n \to \tau_2'}} h_{mn}(\varphi;x_m,y_n) .$$

This theorem is the two dimensional analogue of (2.6).

Proof

By Theorem 4, if $\tau_1 < \infty$, $\tau_2 < \infty$, then

$$h_{m,n}(\varphi;x,y) \to \int_{o,o}^{1,1} \int_{o,o}^{\tau_2 v,\tau_1 u} \frac{\sin s}{s} \frac{\sin t}{t} \, dsdtd^2 g(u,v)$$

$$= \int_{o,o}^{1,1} f(u,v) \, d^2 g(u,v) .$$

Integrating by parts [14, p. 38],

$$h_{m,n}(\varphi;x,y) \to fg \Big\}_{o,o}^{1,1} - \int_o^1 \left\{\frac{df}{du} g \right\}_{v=o}^{1} du - \int_o^1 \left\{ \frac{df}{dv} g \right\}_{u=o}^{1}$$

$$+ \int_{o,o}^{1,1} \frac{d^2 f}{dudv} g \, dudv$$

$$= \int_{0,0}^{1,1} \frac{\sin \tau_1 u}{u} \frac{\sin \tau_2 v}{v} \, dudv - \int_{0,0}^{1,1} \frac{\sin \tau_1 u}{u} \frac{\sin \tau_2 v}{v} \, g(u,1) dudv$$

$$- \int_{0,0}^{1,1} \frac{\sin \tau_1 u}{u} \frac{\sin \tau_2 v}{v} \, g(1,v) dudv + \int_{0,0}^{1,1} \frac{\sin \tau_1 u}{u} \frac{\sin \tau_2 v}{v} \, g(u,v) dudv$$

$$= \int_{0,0}^{1,1} \frac{\sin \tau_1 u}{u} \frac{\sin \tau_2 v}{v} \, g(1,1;u,v) dudv \quad .$$

This completes the proof of Theorem 5 .

We combine Theorem 1 of Chapter 1, Theorems 1 and 5 of the present chapter, and the theorem due to Szász given in (2.6), and obtain the following

(2.44) Theorem 6

If $f(x,y)$ is a 2π-periodic function of bounded variation in the sense of Hardy-Krause, with a discontinuity at the origin, and if $h_{m,n}(f;x,y)$ denotes the mn-th regular Hausdorff transform of the sequence of partial sums of the Fourier series of $f(x,y)$, relative to the Hausdorff weight function $g(u,v)$, then

$$\limsup_{\substack{m,n \to \infty \\ x,y \to 0}} h_{m,n}(f;x,y) = \max_{\tau_1,\tau_2} \left\{ \frac{c}{\pi^2} \int_{0,0}^{1,1} \frac{\sin \tau_1 u}{u} \frac{\sin \tau_2 v}{v} \, g(1,1;u,v) dudv \right.$$

$$+ g_1(o) \int_o^1 \frac{\sin \tau_1 u}{u} \{g(1,1)-g(u,1)\}du + g_2(o) \int_o^1 \frac{\sin \tau_2 v}{v} \cdot$$

$$\cdot \{g(1,1)-g(1,v)\}dv \Big\} ,$$

where c , $g_1(y)$ and $g_2(x)$ are defined in Theorem 1 of Chapter 1.

(2.45) Remarks

In general, τ_1 and τ_2 cannot be restricted to positive values in the statement of Theorem 6.

CHAPTER 3

The Fourier-Stieltjes Transforms in Two Dimensions

§ 1.

Our main object in this chapter is the proof of Theorem 6, which we will need in Chapter 4. This theorem is the two dimensional analogue of the following result due to Lorch and Newman [8, p. 288]:

(3.1) Theorem

If $g(u)$ is continuous and of bounded variation, $0 \leq u \leq 1$, then

$$\int_1^m \left| \int_0^1 \frac{\sin su}{s} \, dg(u) \right| ds = o\,(\log m), \quad m \to \infty \quad .$$

They remark that the converse is also true. If the above equality is satisfied, then $g(u)$ is continuous, and of bounded variation. However in stating the converse, one must make an exception at the origin. For $g(u)$ may have a mass point at the origin and be otherwise continuous, and the above equality will remain satisfied. As we shall see, the converse of Theorem 6 is also true if we permit $g(u,v)$ to have mass points along the coordinate axes.

We begin by giving a development of a few basic properties of the Fourier-Stieltjes transforms in two dimensions. This development follows closely that given by Zygmund [15, Ch. XVI] for the one dimensional case. We could have used some of the results of Bochner and Chandrasekharen [3] for the multi-dimensional case, except that, in some sense, their results are obtained under a slightly stronger hypothesis. To preserve some symmetry, we will, without further explicit mention, deal exclusively with the modified transforms.

(3.2) Definition

Let $g(u,v)$ be of bounded variation in $R \times R$, where R is the real line. The function

$$\varphi(s,t) = \frac{1}{2\pi} \int_{-\infty,-\infty}^{\infty,\infty} e^{-i(su+tv)} d^2 g(u,v)$$

is called the Fourier-Stieltjes transform of $g(u,v)$, or the Fourier transform of $d^2 g(u,v)$.

(3.3) Lemma 1

If $g(u,v)$ is of bounded variation in $R \times R$, then $\varphi(s,t)$ exists and is bounded for all s,t.

Proof

The lemma follows the observation that

$$|\varphi(s,t)| \leq \frac{1}{2\pi} \int_{-\infty,-\infty}^{\infty,\infty} |e^{-i(su+tv)}| \cdot |d^2 g(u,v)| = \frac{1}{2\pi} V(g) .$$

(3.4) Theorem 1

If $g(u,v)$ is of bounded variation, then for arbitrary, real fixed numbers p,q , $g(u+p,v+q;u,v)$ is absolutely integrable.

Proof

Since $g(u,v)$ is of bounded variation, it can be expressed as the difference of two bounded, positively monotonic functions. The problem then reduces to proving the theorem for the case where $g(u,v)$ is already positively monotonic, so that for $p,q > 0$, $g(u+p,v+q;u,v)$ is non-negative. Let

$$\begin{aligned} f(w_1,w_2) &= \int_{-w_2,-w_1}^{w_2,w_1} g(u+p,v+q;u,v)dudv \\ &= \int_{-w_2,-w_1}^{w_2,w_1} \{g(u+p,v+q) - g(u+p,v) - g(u,v+q) + g(u,v)\}dudv \\ &= \left\{ \int_{-w_2+q,-w_1+p}^{w_2+q,w_1+p} - \int_{-w_2,-w_1+p}^{w_2,w_1+p} - \int_{-w_2+q,-w_1}^{w_2+q,w_1} + \int_{-w_2,-w_1}^{w_2,w_1} \right\} g(u,v)dudv \\ &= \left\{ \int_{w_2,w_1}^{w_2+q,w_1+p} - \int_{-w_2,-w_1}^{w_2+q,-w_1+p} - \int_{-w_2,w_1}^{-w_2+q,u_1+p} + \int_{-w_2,-w_1}^{-w_2+q,-w_1+p} \right\} g(u,v)dudv . \end{aligned}$$

Now let $w_1, w_2 \to \infty$. Then $g(u,v)$ tends, respectively to $g(\infty,\infty)$,

$g(-\infty,\infty)$, $g(\infty,-\infty)$ and $g(-\infty,-\infty)$, in each of the four regions of integration in the last expression. Then

$$\lim_{w_1,w_2 \to \infty} f(w_1,w_2) = pqg(\infty,\infty;-\infty,-\infty)$$

$$= pqV(g) \quad ,$$

since $g(u,v)$ is positively monotonic. This proves the absolute integrability of $g(u+p,v+q;u,v)$.

(3.5) Remarks

It is also true that for each fixed u , resp. fixed v , $g(u+p,v+q;u,v)$ is an absolutely integrable function of v alone, resp. of u alone. The proof is an obvious simplification of the one given above. Assuming that $g(u,v)$ is positively monotonic, the integrals are, respectively, $qg(u+p,\infty;u,-\infty)$ and $pg(\infty,v+q;-\infty,v)$, where $p,q > 0$.

(3.6) Theorem 2

If $g(u,v)$ is a normalized function of bounded variation, then

$$g(u+p,v+q;u,v) = \frac{1}{2\pi}\int_{-\infty,-\infty}^{\infty,\infty} \varphi(s,t)\,\frac{\{e^{ips}-1\}}{is}\,\frac{\{e^{iqt}-1\}}{it}\;e^{i(su+tv)}dsdt$$

everywhere.

Proof

For fixed p,q, we have

$$\varphi(s,t)e^{i(ps+qt)} = \frac{1}{2\pi}\int_{-\infty,-\infty}^{\infty,\infty} e^{-i\{s(u-p)+t(v-q)\}}d^2g(u,v)$$

$$= \frac{1}{2\pi}\int_{-\infty,-\infty}^{\infty,\infty} e^{-i(su+tv)}d^2g(u+p,v+q) \quad ,$$

where we have set $u-p = u'$, $v-q = v'$, and dropped the primes. Similarly,

$$-\varphi(s,t)e^{ips} = -\frac{1}{2\pi}\int_{-\infty,-\infty}^{\infty,\infty} e^{-i(su+tv)}d^2g(u+p,v)$$

$$-\varphi(s,t)e^{iqt} = -\frac{1}{2\pi}\int_{-\infty,-\infty}^{\infty,\infty} e^{-i(su+tv)}d^2g(u,v+q)$$

$$\varphi(s,t) = \frac{1}{2\pi}\int_{-\infty,-\infty}^{\infty,\infty} e^{-i(su+tv)}d^2g(u,v) \ .$$

Then

$$\varphi(s,t)\{e^{ips}-1\}\{e^{iqt}-1\} = \frac{1}{2\pi}\int_{-\infty,-\infty}^{\infty,\infty} e^{-i(su+tv)}d^2g(u+p,v+q;u,v) \quad .$$

Since $g(u,v)$ is of bounded variation, $g(u+p,v+q;u,v) \to 0$ as $u,v \to \pm\infty$ independently. Integration by parts [14, p. 38] yields

$$\varphi(s,t)\{e^{ips}-1\}\{e^{iqt}-1\} = \frac{1}{2\pi}\, e^{-i(su+tv)} g(u+p,v+q;u,v) \Big\}_{-\infty,-\infty}^{\infty,\infty}$$

$$+ \frac{is}{2\pi} \int_{-\infty}^{\infty} \{e^{-i(su+tv)} g(u+p,v+q;u,v)\}_{v=-\infty}^{\infty}\, du$$

$$+ \frac{it}{2\pi} \int_{-\infty}^{\infty} \{e^{-i(su+tv)} g(u+p,v+q;u,v)\}_{u=-\infty}^{\infty}\, dv$$

$$+ \frac{is\cdot it}{2\pi} \int_{-\infty,-\infty}^{\infty,\infty} e^{-i(su+tv)} g(u+p,v+q;u,v)\,du\,dv$$

$$= \frac{is\cdot it}{2\pi} \int_{-\infty,-\infty}^{\infty,\infty} e^{-i(su+tv)} g(u+p,v+q;u,v)\,du\,dv \quad ,$$

where the last equality follows by the preceding remark. Then

$$(3.7) \qquad \varphi(s,t)\,\frac{\{e^{ips}-1\}}{is}\,\frac{\{e^{iqt}-1\}}{it}$$

is the Fourier transform of $g(u+p,v+q;u,v)$, an absolutely integrable function of bounded variation in $R \times R$. We set $G(u,v) = g(u+p,v+q;u,v)$, and let

$$\psi_{w_1,w_2}(u,v) = \frac{1}{2\pi} \int_{-w_2,-w_1}^{w_2,w_1} \varphi(s,t)\,\frac{\{e^{ips}-1\}}{is}\,\frac{\{e^{iqt}-1\}}{it}\, e^{i(su+tv)}\,ds\,dt$$

$$= \frac{1}{(2\pi)^2} \int_{-w_2,w_1}^{w_2,w_1} \int_{-\infty,-\infty}^{\infty,\infty} G(x,y) e^{-i(sx+ty)}\,dx\,dy\; e^{i(su+tv)}\,ds\,dt$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty,-\infty}^{\infty,\infty} G(x,y) \int_{-w_2,-w_1}^{w_2,w_1} e^{i\{s(u-x)+t(v-y)\}} dsdtdxdy$$

$$= \frac{1}{\pi^2} \int_{-\infty,-\infty}^{\infty,\infty} G(x,y) \frac{\sin w_1(x-u)}{x-u} \frac{\sin w_2(y-v)}{y-v} dxdy$$

$$= \frac{1}{\pi^2} \int_{-\infty,-\infty}^{\infty,\infty} G(u+x,v+y) \frac{\sin w_1 x}{x} \frac{\sin w_2 y}{y} dxdy$$

$$= \frac{1}{\pi^2} \int_{o,o}^{\infty,\infty} \{G(u+x,v+y) + G(u-x,v+y) + G(u+x,v-y) + G(u-x,v-y)\}$$

$$\frac{\sin w_1 x}{x} \frac{\sin w_2 y}{y} dxdy \quad ,$$

where the change in the order of integration is justified by Fubini's theorem. Making the obvious substitution, we have

$$(3.8) \qquad \psi_{w_1,w_2}(u,v) = \frac{1}{\pi^2} \int_{o,o}^{\infty,\infty} G_{u,v}(x,y) \frac{\sin w_1 x}{x} \frac{\sin w_2 y}{y} dxdy \quad .$$

To complete the proof of Theorem 2, we need a lemma.

(3.9) Lemma

If $f(x,y)$ is of bounded variation and absolutely integrable in the cell $(\infty,\infty;\ o,o]$, then

$$\lim_{w_1,w_2\to\infty}\int_{0,0}^{\infty,\infty} f(x,y)\,\frac{\sin w_1x}{x}\,\frac{\sin w_2y}{y}\,dxdy = \frac{\pi^2}{4}\,f(0^+,0^+) \quad .$$

(3.10) Remarks

For the case where the upper limit of integration with respect to each variable is a constant c , $0 < c \leq \pi/2$, and where w_1, w_2 are restricted to vary over the positive integers, a proof of (3.9) is given by Hobson [6, p. 705]. The extension of his proof to a proof of the lemma as stated here involves only the obvious modifications, and we will therefore omit it.

Now $G_{u,v}(x,y)$, as a finite sum of absolutely integrable functions of bounded variation, is itself absolutely integrable and of bounded variation. Hence the conditions of (3.9) are satisfied, and so, by (3.8),

$$\begin{aligned}\lim_{w_1,w_2\to\infty} \psi_{w_1,w_2}(u,v) &= \tfrac{1}{4}\,G(0^+,0^+)\\ &= \tfrac{1}{4}\,\{G(u^+,v^+) + G(u^+,v^-) + G(u^-,v^+) + G(u^-,v^-)\}\\ &= G(u,v)\\ &= g(u+p,u+q;u,v) \quad .\end{aligned}$$

Here we have used the observation that $G(u,v)$ is a finite sum of normalized functions, and so is normalized. This completes the proof of Theorem 2.

(3.11) Corollary

If $g(u,v)$ is a normalized function of bounded variation in the cell $(\infty,\infty;-\infty,-\infty)$, then

$$g(u,v;o,o) = \frac{1}{2\pi} \int_{-\infty,-\infty}^{\infty,\infty} \varphi(s,t) \frac{\{e^{isu}-1\}}{is} \frac{\{e^{itv}-1\}}{it} \, dsdt$$

Proof

In Theorem 2, set $u = v = 0$, then $p = u$ and $q = v$.

(3.12) Theorem 3

Let $g(u,v)$ be a normalized function of bounded variation in $R \times R$. Then

$$g(u+p,v+q;u-p,v-q) = \frac{2}{\pi} \int_{-\infty,-\infty}^{\infty,\infty} \varphi(s,t) \frac{\sin ps}{s} \frac{\sin qt}{t} e^{i(su+tv)} dsdt \ .$$

Proof

By (3.11) ,

$$g(u\pm p,v\pm q;o,o) = \frac{1}{2\pi} \int_{-\infty,-\infty}^{\infty,\infty} \varphi(s,t) \frac{\{e^{is(u\pm p)}-1\}}{is} \frac{\{e^{it(v\pm q)}-1\}}{it} \, dsdt \ .$$

Then

$$g(u+p,v+q;o,o) - g(u+p,v-q;o,o) - g(u-p,v+q;o,o) + g(u-p,v-q;o,o)$$

$$= g(u+p,v+q) - g(u+p,v-q) - g(u-p,v+q) + g(u-p,v-q)$$

$$= g(u+p,v+q;u-p,v-q)$$

$$= \frac{1}{2\pi}\int_{-\infty,-\infty}^{\infty,\infty} \varphi(s,t) \frac{\{e^{is(u+p)}-e^{is(u-p)}\}}{is} \frac{\{e^{it(v+q)}-e^{it(v-q)}\}}{it} dsdt$$

$$= \frac{1}{2\pi}\int_{-\infty,-\infty}^{\infty,\infty} \varphi(s,t) \frac{\{e^{ips}-e^{-ips}\}}{is} \frac{\{e^{iqt}-e^{-iqt}\}}{it} e^{i(su+tv)}dsdt$$

$$= \frac{2}{\pi}\int_{-\infty,-\infty}^{\infty,\infty} \varphi(s,t) \frac{\sin ps}{s} \frac{\sin qt}{t} e^{i(su+tv)}dsdt$$

This completes the proof of Theorem 3.

(3.13) Theorem 4

Under the hypothesis of Theorem 3,

$$\int_{-\infty,-\infty}^{\infty,\infty} |g(u+p,v+q;u-p,v-q)|^2 dudv$$

$$= 16\int_{-\infty,-\infty}^{\infty,\infty} |\varphi(s,t)|^2 \frac{\sin^2 ps}{s^2} \frac{\sin^2 qt}{t^2} dsdt$$

Proof

By Parseval's relation [3, p. 67] , if

$$g(u,v) = \frac{1}{2\pi} \int_{-\infty,-\infty}^{\infty,\infty} \varphi(s,t)\, e^{i(su+tv)} ds dt \quad ,$$

then

$$\int_{-\infty,-\infty}^{\infty,\infty} |\varphi(s,t)|^2 ds dt = \int_{-\infty,-\infty}^{\infty,\infty} |g(u,v)|^2 du dv \quad .$$

Theorem 4 follows immediately from this relation if, by Theorem 3, we replace $g(u,v)$ by $g(u+p,v+q;u-p,v-q)$, and $\varphi(s,t)$ by

$$4\varphi(s,t)\, \frac{\sin ps}{s}\, \frac{\sin qt}{t} \quad .$$

(3.14) Theorem 5

$$\int_{-\infty,-\infty}^{\infty,\infty} \Big| \int_{-\infty,-\infty}^{\infty,\infty} \frac{\sin su}{s}\, \frac{\sin tv}{t}\, d^2 g(u,v) \Big|^2 ds dt$$

$$\leq \int_{-\infty,-\infty}^{\infty,\infty} \Big| \int_{-\infty,-\infty}^{\infty,\infty} \frac{e^{-i(su+tv)}}{st}\, d^2 g(u,v) \Big|^2 ds dt \quad .$$

Proof

If the right side is infinite, the inequality is obvious. Suppose, then, that the right side exists. Then

$$\left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{\sin su}{s} \; \frac{\sin tv}{t} \, d^2 g(u,v) \right| = \frac{1}{4} \left| \int_{-\infty,-\infty}^{\infty,\infty} \{ e^{i(su+tv)} \right.$$

$$\left. - e^{i(su+tv)} - e^{i(su-tv)} - e^{-i(su-tv)} \} \frac{d^2 g(u,v)}{st} \right|$$

$$\leq \frac{1}{2} \left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{e^{-i(su+tv)}}{st} \, d^2 g(u,v) \right|$$

$$+ \frac{1}{2} \left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{e^{-i(su-tv)}}{st} \, d^2 g(u,v) \right|$$

$$= a + b \quad ,$$

say. Then

$$\left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{\sin su}{s} \, \frac{\sin tv}{t} \, d^2 g(u,v) \right|^2 \leq (a+b)^2 \leq 2a^2 + 2b^2$$

$$= \frac{1}{2} \left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{e^{-i(su+tv)}}{st} \, d^2 g(u,v) \right|^2$$

$$+ \frac{1}{2} \left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{e^{-i(su-tv)}}{st} \, d^2 g(u,v) \right|^2$$

and

$$(3.15) \quad \int_{-\infty,-\infty}^{\infty,\infty} \left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{\sin su}{s} \frac{\sin tv}{t} d^2g(u,v) \right|^2 dsdt$$

$$\leq \frac{1}{2} \int_{-\infty,-\infty}^{\infty,\infty} \left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{e^{-i(su+tv)}}{st} d^2g(u,v) \right|^2 dsdt$$

$$+ \frac{1}{2} \int_{-\infty,-\infty}^{\infty,\infty} \left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{e^{-i(su-tv)}}{st} d^2g(u,v) \right|^2 dsdt \quad .$$

In the last integral we let $t = -t'$, $dt = -dt'$. Then, dropping the primes,

$$(3.16) \quad \int_{-\infty,-\infty}^{\infty,\infty} \left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{\sin su}{s} \frac{\sin tv}{t} d^2g(u,v) \right|^2 dsdt$$

$$\leq \frac{1}{2} \int_{-\infty,-\infty}^{\infty,\infty} \left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{e^{-i(su+tv)}}{st} d^2g(u,v) \right|^2 dsdt$$

$$- \frac{1}{2} \int_{\infty,-\infty}^{-\infty,\infty} \left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{e^{-i(su+tv)}}{st} d^2g(u,v) \right|^2 dsdt$$

$$= \frac{1}{2} \int_{-\infty,-\infty}^{\infty,\infty} \left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{e^{-i(su+tv)}}{st} d^2g(u,v) \right|^2 dsdt$$

$$+ \frac{1}{2} \int_{-\infty,-\infty}^{\infty,\infty} \left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{e^{-i(su+tv)}}{st} d^2g(u,v) \right|^2 dsdt$$

$$= \int_{-\infty,-\infty}^{\infty,\infty} \left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{e^{-i(su+tv)}}{st} d^2g(u,v) \right|^2 dsdt \quad .$$

This complets the proof of Theorem 5.

(3.17) Remarks

The only place where we used the limits of integration in the proof of the above theorem is in (3.16), where we interchanged the upper and lower limits of integration with respect to t. It follows that the theorem is true if any or all of the limits of integration are taken finite, provided that the limits of integration with respect to t are symmetric about the origin, and hence provided that the limits of integration with respect to one of s or t, or both, are symmetric about the origin.

In the one dimensional case we also have, trivially,

$$(3.18) \qquad \left| \int_{-\infty}^{\infty} \sin su\, dg(u) \right| \le \left| \int_{-\infty}^{\infty} e^{-isu} dg(u) \right|$$

for every value of s. In the two dimensional case, it is equally easy to prove that

$$\left| \int_{-\infty,-\infty}^{\infty,\infty} \sin su \sin tv\, d^2g(u,v) \right| \le \left| \int_{-\infty,-\infty}^{\infty,\infty} e^{-isu} \sin tv\, d^2g(u,v) \right|$$

for each s, t. It is also true that if $g(u,v)$ factors, so that $g(u,v) = g_1(u) \cdot g_2(v)$, then

$$(3.19)\qquad \left| \int_{-\infty,-\infty}^{\infty,\infty} \sin su \sin tv\, d^2 g(u,v) \right| \le \left| \int_{-\infty,-\infty}^{\infty,\infty} e^{-i(su+tv)} d^2 g(u,v) \right| ,$$

for then the proof of (3.19) reduces to a proof of the inequality (3.18). However, (3.19) does not hold in general. For let $s = t = 1$ and

$$g(u,v) = 0 , \quad \begin{array}{ll} u < o , & v < o \\ u < o , & v \ge o \\ u \ge o , & v < o \end{array}$$

$$g(u,v) = 1 , \quad \begin{cases} o \le u < \frac{\pi}{2} , & o \le v \\ o \le u , & o \le v < \frac{\pi}{2} \end{cases}$$

$$g(u,v) = 2 , \quad \frac{\pi}{2} \le u,\ v,$$

so that $d^2 g(u,v) = 1$ when $(u,v) = (o,o)$ and when $(u,v) = (\pi/2, \pi/2)$, and $d^2 g(u,v) = 0$ otherwise. Then the left side of the inequality (3.19) equals 1 , while the right side equals 0 .

§ 2.

We are now ready to prove the two dimensional analogue of the theorem given in (3.1).

(3.20) Theorem 6

If $g(u,v)$ is of bounded variation in the cell $[1,1;0,0]$, and has no mass points bounded away from the coordinate axes, then

$$\int_{1,1}^{n,m} \left| \int_{o,o}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} d^2 g(u,v) \right| dsdt = o\ (\log m \log n)\ , \quad m,n \to \infty \quad .$$

Proof

Since mass points along the axes will clearly not contribute anything to the integral in the theorem, we may assume that $g(u,v)$ has no mass points. We extend $g(u,v)$ to the entire plane as follows:

$$\begin{array}{rlll} g(u,v) & = g(1,v)\ , & 1 \leq u\ , & o \leq v \leq 1 \\ & = g(u,1)\ , & 1 \leq v\ , & o \leq u \leq 1 \\ & = g(o,v)\ , & u \leq o\ , & o \leq v \leq 1 \\ & = g(u,o)\ , & v \leq o\ , & o \leq u \leq 1 \\ & = g(o,o)\ , & u,v \leq o & \\ & = g(1,o)\ , & u \leq o\ , \ u \geq 1 & \\ & = g(o,1) & u \leq o\ , \ v \geq 1 & \\ & = g(1,1) & 1 \leq u,v\ \ . & \end{array}$$

The function $g(u,v)$, extended in this manner, has the same total variation in the entire plane as the original function has in the unit cell. Also, since $g(u_2,v_2;u_1,v_1) = 0$ whenever the cell $[u_2,v_2;u_1,v_1]$ does not intersect the cell $[1,1;0,0]$, it follows that

$$\int_{0,0}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} d^2g(u,v) = \int_{-\infty,-\infty}^{\infty,\infty} \frac{\sin su}{s} \frac{\sin tv}{t} d^2g(u,v) \quad .$$

Now let integers h, k be determined by the inequalities

$$2^h < m \leq 2^{h+1} , \qquad 2^k < n \leq 2^{k+1} \quad .$$

Then

$$\int_{1,1}^{n,m} \Big| \int_{0,0}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} d^2g(u,v) \Big| dsdt$$

$$= \int_{1,1}^{n,m} \Big| \int_{-\infty,-\infty}^{\infty,\infty} \frac{\sin su}{s} \frac{\sin tv}{t} d^2g(u,v) \Big| dsdt$$

$$\leq \sum_{i,j=0}^{h,k} \int_{2^j,2^i}^{2^{j+1},2^{i+1}} \Big| \int_{-\infty,-\infty}^{\infty,\infty} \frac{\sin su}{s} \frac{\sin tv}{t} d^2g(u,v) \Big| dsdt$$

$$= \sum_{i,j=0}^{h,k} \Big\{ \Big\{ \int_{2^j,2^i}^{2^{j+1},2^{i+1}} \Big| \int_{-\infty,-\infty}^{\infty,\infty} \frac{\sin su}{s} \frac{\sin tv}{t} d^2g(u,v) \Big| dsdt \Big\}^2 \Big\}^{1/2}$$

$$\leq \sum_{i,j=0}^{h,k} \Big\{ \int_{2^j}^{2^{j+1}} dt \int_{2^i}^{2^{i+1}} \Big\{ \int_{2^i}^{2^{i+1}} \Big| \int_{-\infty,-\infty}^{\infty,\infty} \frac{\sin su}{s} \frac{\sin tv}{t} d^2g(u,v) ds \Big\}^2 d$$

$$\leq \sum_{i,j=0}^{h,k} \Big\{ 2^j \int_{2^j}^{2^{j+1}} \Big\{ \int_{2^i}^{2^{i+1}} \Big| \int_{-\infty,-\infty}^{\infty,\infty} \frac{\sin su}{s} \frac{\sin tv}{t} d^2g(u,v) \Big|^2 dsdt \Big\}^{1/2}$$

$$= \sum_{i,j=0}^{h,k} \Big\{ 2^{i+j} \int_{2^j,2^i}^{2^{j+1},2^{i+1}} \Big| \int_{-\infty,-\infty}^{\infty,\infty} \frac{\sin su}{s} \frac{\sin tv}{t} d^2g(u,v) \Big|^2 dsdt \Big\}^{1/2}$$

by a repeated application of the Cauchy-Schwartz inequality for integrals. Then by (3.15)

$$\int_{1,1}^{m,m} \left| \int_{0,0}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} \, d^2 g(u,v) \right| dsdt$$

$$\leq \frac{1}{\sqrt{2}} \sum_{i,j=0}^{h,k} \Big\{ 2^{i+j} \int_{2^j,2^i}^{2^{j+1},2^{i+1}} \Big\{ \left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{e^{-i(su+tv)}}{st} \, d^2 g(u,v) \right|^2$$

$$+ \left| \int_{-\infty,-\infty}^{\infty,\infty} \frac{e^{-i(su-tv)}}{st} \, d^2 g(u,v) \right|^2 \Big\} \, dsdt \Big\}^{1/2}$$

$$= \sqrt{2}\,\pi \sum_{i,j=0}^{h,j} \Big\{ 2^{i+j} \int_{2^j,2^i}^{2^{j+1},2^{i+1}} \Big\{ \left| \frac{\varphi(s,t)}{st} \right|^2 + \left| \frac{\varphi(s,-t)}{st} \right|^2 \Big\} dsdt \Big\}^{1/2}$$

$$\leq 2\pi \sum_{i,j=0}^{h,k} \Big\{ 2^{i+j} \int_{2^j,2^i}^{2^{j+1},2^{i+1}} \frac{\sin^2 \frac{s}{2^i}}{s^2} \, \frac{\sin^2 \frac{t}{2^j}}{t^2} \Big\{ |\varphi(s,t)|^2 + |\varphi(s,-t)|^2 dsdt \Big\}^{1/2}$$

$$\leq \frac{\pi}{2} \sum_{i,j=0}^{h,k} \Big\{ 2^{i+j} \cdot 16 \int_{-\infty,-\infty}^{\infty,\infty} \frac{\sin^2 \frac{s}{2^i}}{s^2} \, \frac{\sin^2 \frac{t}{2^j}}{t^2} \, |\varphi(s,t)|^2 dsdt \Big\}^{1/2}$$

$$= \frac{\pi}{2} \sum_{i,j=0}^{h,k} \Big\{ 2^{i+j} \int_{-\infty,-\infty}^{\infty,\infty} \left| g\left(u+\frac{1}{2^i}, \; v+\frac{1}{2^j} \; ; \; u-\frac{1}{2^i}, \; v-\frac{1}{2^j}\right) \right|^2 dudv \Big\}^{1/2},$$

where we have used the defining relation (3.2), and then Theorem 4. Next, we define the modulus of continuity for a function of two variables.

(3.21) Definition

Let $g(u,v)$ be defined in some domain D in $R \times R$. By the modulus of continuity in the sense of Hardy-Krause of the function $g(u,v)$, we shall mean the quantity

$$\omega(g;\delta_1,\delta_2) = \sup \{ |g(u+\delta_1,v+\delta_2;u-\delta_1,v-\delta_2)| : (u\pm\delta_1,v\pm\delta_2) \in D \} .$$

(3.22) Remarks

It is clear that if $g(u,v)$ is continuous, then $\omega(g;\delta_1,\delta_2) \to 0$ as $\delta_1,\delta_2 \to 0$ or if only $\delta_1 \to 0$ or $\delta_2 \to 0$ while the other one is fixed. This convergence is clearly monotonic: $\alpha_1 \geq \beta_1$, $\alpha_2 \geq \beta_2$ implies $\omega(g;\alpha_1,\alpha_2) \geq \omega(g;\beta_1,\beta_2)$.

Note that the modulus of continuity, so defined, does not distinguish between continuous functions and a certain class of functions which are discontinuous in the usual sense. Thus if $g(u,v) = -v$, $u < 0$, $g(u,v) = v$, $u > 0$, and $g(u,v) = 0$ when $u = o$, then $\omega(g;\delta_1,\delta_2) \to 0$ as $\delta_1,\delta_2 \to 0$, but $g(u,v)$ is certainly not continuous in the usual sense.

We return to the proof of Theorem 6. We have

$$\int_{1,1}^{n,m} \left| \int_{o,o}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} \, d^2 g(u,v) \right| dsdt$$

$$\leq \frac{\pi}{2} \sum_{i,j=o}^{h,k} \left\{ 2^{i+j} \int_{-\infty,-\infty}^{\infty,\infty} |g(u+2^{-i}, v+2^{-j}; u-2^{-i}, v-2^{-j})|^2 dudv \right\}^{1/2}$$

$$= \frac{\pi}{2} \sum_{i,j=o}^{h,k} \left\{ \int_{-\infty,-\infty}^{\infty,\infty} \sum_{p,q=1}^{2^i,2^j} |g(u+\frac{2p-1}{2^i}, v+\frac{2q-1}{2^j} ; u+\frac{2p-3}{2^i}, v+\frac{2q-3}{2^j})|^2 dudv \right\}^{1/2}$$

$$= \frac{\pi}{2} \sum_{i,j=o}^{h,k} \left\{ \int_{-1,-1}^{2,2} \sum_{p,q=1}^{2^i,2^j} |g(u+\frac{2p-1}{2^i}, v+\frac{2q-1}{2^j} ; u+\frac{2p-3}{2^i}, v+\frac{2q-3}{2^j})|^2 dudv \right\}^{1/2}$$

$$\leq \frac{\pi}{2} \sum_{i,j=o}^{h,k} \left\{ 9\omega(g;2^{-i},2^{-j}) \cdot V(g) \right\}^{1/2}$$

$$\leq 2\pi \left\{ V(g) \right\}^{1/2} \sum_{i,j=o}^{h,k} \left\{ \omega(g;2^{-i},2^{-j}) \right\}^{1/2} \quad ,$$

where we used the observation that in the range of summation,

$$g(u + \frac{2p-1}{2^i} , v + \frac{2q-1}{2^j} ; u + \frac{2p-3}{2^i} , v + \frac{2q-3}{2^j}) = 0$$

whenever $(u,v) \notin [2,2;-1,-1]$.

We assume that $\omega(g;2^{-i},2^{-j}) \to 0$ as $i,j \to \infty$ and take m, n

through the sequences $1,2,4,8,\ldots,2^{h+1}$, $1,2,4,8,\ldots,2^{k+1}$, respectively. Then, given $\epsilon > 0$, for h,k large enough there exist integers $h_1 < h$, $k_1 < k$ such that

$$\{\omega(g;2^{-i},2^{-j})\}^{1/2} < \epsilon , \quad i > h_1 , \quad j > k_1 \quad .$$

Then

$$\int_{1,1}^{2^{k+1},2^{h+1}} \left| \int_{o,o}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} \, d^2 g(u,v) \right| dsdt$$

$$\leq 2\pi \, \{V(g)\}^{1/2} \sum_{i,j=o}^{h,k} \{\omega(g;2^{-i},2^{-j})\}^{1/2}$$

$$= 2\pi \, \{V(g)\}^{1/2} \left\{ \sum_{j=o}^{k_1} \sum_{i=o}^{h_1} + \sum_{j=k_1+1}^{k} \sum_{i=o}^{h_1} + \sum_{j=o}^{k_1} \sum_{i=h_1+1}^{h} + \sum_{j=k_1+1}^{k} \sum_{i=h_1+1}^{h} \right\}$$

$$\leq 2\pi\{V(g)\}^{1/2} \, \{(h_1+1)(k_1+1)\{V(g)\}^{1/2} + (h_1+1)(k-k_1)\{V(g)\}^{1/2}$$

$$+ (h-h_1)(k_1+1)\{V(g)\}^{1/2} + \epsilon(h-h_1)(k-k_1)\{V(g)\}^{1/2} \}$$

$$= O(1)\{O(1) + O(k) + O(h) + O(\epsilon hk)\}$$

$$= o(hk) + O(\epsilon hk)$$

$$= o(hk) = o(\log m \log n) , \quad m,n \to \infty ,$$

since ϵ is arbitrary. This proves Theorem 6.

(3.23) Theorem

If $g(u,v)$ is defined on the cell $[1,1;0,0]$, and if

$$\int_{1,1}^{n,m} \left| \int_{o,o}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} \, d^2 g(u,v) \right| ds dt = o(\log m \log n), \quad m,n \to \infty ,$$

then $g(u,v)$ is of bounded variation and has no mass points bounded away from the coordinate axes.

Proof

The bounded variation is trivial. That $g(u,v)$ may have mass points along the coordinate axes and yet the equality holds is just as easy to show by example. The remainder of the theorem follows from Theorems 2 and 3 of the next chapter, since $M\{f(s,t)\} = 0$ implies $f(s,t) \equiv 0$, and this in turn implies that $g(u,v)$ has no mass points bounded away from the coordinate axes.

CHAPTER 4

The Lebesgue Constants for the Hausdorff Means of Double Fourier Series

§ 1.

The Lebesgue constants for double Fourier series are defined by

$$(4.1) \qquad L(m,n) = \frac{4}{\pi^2} \int_{0,0}^{\pi,\pi} \left| \frac{\sin (m+\frac{1}{2})s}{2 \sin \frac{1}{2} s} \, \frac{\sin (n+\frac{1}{2})t}{2 \sin \frac{1}{2} t} \right| dsdt$$

$$= \frac{4}{\pi^2} \int_{0,0}^{\pi,\pi} |D_{m,n}(s,t)| dsdt$$

where

$$(4.2) \qquad D_{m,n}(s,t) = \frac{\sin (m+\frac{1}{2})s}{2 \sin \frac{1}{2} s} \cdot \frac{\sin (n+\frac{1}{2})t}{2 \sin \frac{1}{2} t}$$

is the Dirichlet kernel for double Fourier series. It is easy to show, in precisely the same manner as in the one dimensional case, that the divergence of these constants implies the existence of a continuous, periodic function of two variables whose Fourier series diverges at a point, and also of another such function whose Fourier series converges everywhere, but not uniformly in the neighborhood of some point. We call these the du Bois-Reymond singularity and the Lebesgue singularity respectively.

If the sequence $\{D_{k,\ell}(s,t)\}$ is transformed by some summability method and we denote the mn-th transform by $K_{m,n}(s,t)$, then the sequence of constants defined by

$$\frac{4}{\pi^2}\int_{0,0}^{\pi,\pi} |K_{m,n}(s,t)|dsdt$$

are the Lebesgue constants for that summability method. The divergence of these constants implies the existence of a periodic, continuous function whose Fourier series cannot be summed by the method at some point, and another such function whose Fourier series is summable everywhere, but not uniformly in the neighborhood of some point. These are, respectively, the du Bois-Reymond and the Lebesgue singularities for the summability method in question.

Lorch and Newman [8] investigated the Lebesgue constants for the regular Hausdorff means in the one dimensional case and proved the following

(4.3) Theorem

Let $L(m;g)$ denote the m-th Lebesgue constant for the regular Hausdorff method with the weight function $g(u)$. Then

$$L(m;g) = c(g)\log m + o(\log m), \quad m \to \infty,$$

where

$$c(g) = \frac{2}{\pi^2}|g(1)-g(1-)| + \frac{1}{\pi} M\left\{\left| \sum_k \left\{g(u_k^+)-g(u_k^-)\right\} \sin u_k x \right|\right\}.$$

Here u_k is the k-th discontinuity of $g(u)$, and the summation extends over all such possibly countably infinite values. $M\{f(x)\}$ represents the mean value of the almost periodic function $f(x)$. Furthermore,

$$0 \leq c(g) \leq \frac{4}{\pi^2} V(g) ,$$

$0 \leq u \leq 1$, and $c(g) = 0$ if and only if $g(u)$ is continuous. If the method is totally regular so that $V(g) = 1$, then $c(g) = 4/\pi^2$ if and only if the method is ordinary convergence. Also, if $g(1) = g(1-)$, then $c(g) \leq 2/\pi^2$, in which case $c(g) = 2/\pi^2$ if and only if the method is of Euler type.

We seek to investigate the Lebesgue constants for the regular Hausdorff means in a similar manner in the two dimensional case. We denote by $K_{m,n}(s,t)$ the mn-th Hausdorff transform of the sequence $\{D_{k,\ell}(s,t)\}$, where $D_{k,\ell}(s,t)$ is defined by (1.2). Then

$$K_{m,n}(s,t) = \sum_{k,\ell=0}^{m,n} \binom{n}{\ell}\binom{m}{k} D_{k,\ell}(s,t) \int_{0,0}^{1,1} u^k(1-u)^{m-k} v^\ell (1-v)^{n-\ell} d^2 g(u,v)$$

$$= \frac{1}{4} \sum_{k,\ell=0}^{m,n} \binom{n}{\ell}\binom{m}{k} \frac{\operatorname{Im} e^{i(k+\frac{1}{2})s}}{\sin \frac{1}{2} s} \, \frac{\operatorname{Im} e^{i(\ell+\frac{1}{2})t}}{\sin \frac{1}{2} t}$$

$$\cdot \int_{0,0}^{1,1} u^k(1-u)^{m-k} v^\ell (1-v)^{n-\ell} d^2 g(u,v)$$

$$= \frac{1}{4} \int_{0,0}^{1,1} \frac{\operatorname{Im} \left\{ (1-u+ue^{is})^m \, e^{i\frac{s}{2}} \right\}}{\sin \frac{1}{2} s} \, \frac{\operatorname{Im} \left\{ (1+v+ve^{it})^n \, e^{i\frac{t}{2}} \right\}}{\sin \frac{1}{2} t} \, d^2 g(u,v)$$

$$= \frac{1}{4} \int_{o,o}^{1,1} \rho_1^m \rho_2^n \frac{\sin(m\alpha+\frac{s}{2})}{\sin\frac{s}{2}} \frac{\sin(n\beta+\frac{t}{2})}{\sin\frac{t}{2}} \; d^2g(u,v)$$

$$= \frac{1}{4} \int_{o,o}^{1,1} \rho_1^m \rho_2^n \sin m\alpha \sin n\beta \cot\frac{s}{2} \cot\frac{t}{2} \, d^2g(u,v)$$

$$+ \frac{1}{4} \int_{o,o}^{1,1} \rho_1^m \rho_2^n \sin m\alpha \cos n\beta \cot\frac{s}{2} \, d^2g(u,v)$$

$$+ \frac{1}{4} \int_{o,o}^{1,1} \rho_1^m \rho_2^n \sin n\beta \cos m\alpha \cot\frac{t}{2} \, d^2g(u,v)$$

$$+ \frac{1}{4} \int_{o,o}^{1,1} \rho_1^m \rho_2^n \cos m\alpha \cos n\beta \, d^2g(u,v)$$

where α, β, ρ_1 and ρ_2 are defined in (2.16). Then

$$(4.4) \qquad K_{m,n}(s,t) = \frac{1}{4} \int_{o,o}^{1,1} \rho_1^m \rho_2^n \sin m\alpha \sin n\beta \left\{\cot\frac{s}{2} - \frac{2}{s} + \frac{2}{s}\right\}$$

$$\cdot \left\{\cot\frac{t}{2} - \frac{2}{t} + \frac{2}{t}\right\} d^2g(u,v)$$

$$+ \frac{1}{4} \int_{o,o}^{1,1} \rho_1^m \rho_2^n \sin m\alpha \cos n\beta \left\{\cot\frac{s}{2} - \frac{2}{s} + \frac{2}{s}\right\} d^2g(u,v)$$

$$+ \frac{1}{4} \int_{o,o}^{1,1} \rho_1^m \rho_2^n \sin n\beta \cos m\alpha \left\{\cot\frac{t}{2} - \frac{2}{t} + \frac{2}{t}\right\} d^2g(u,v)$$

$$+ \frac{1}{4} \int_{o,o}^{1,1} \rho_1^m \rho_2^n \cos m\alpha \cos n\beta \, d^2g(u,v)$$

$$= \int_{o,o}^{1,1} \rho_1^m \rho_2^n \frac{\sin m\alpha}{s} \frac{\sin n\beta}{t} \, d^2g(u,v)$$

$$+\int_{o,o}^{1,1} \rho_1^m \frac{\sin m\alpha}{s}\, \varphi(n;t,v)d^2g(u,v) + \int_{o,o}^{1,1} \rho_2^n \frac{\sin n\beta}{t}\, \varphi(m;s,u)d^2g(u,v)$$

$$+ \int_{o,o}^{1,1} \varphi(m;s,u)\, \varphi(n;t,v)d^2g(u,v)$$

where

$$\varphi(m;s,u) = \frac{1}{2}\rho_1^m \{\sin m\alpha\, \{\cot \frac{s}{2} - \frac{2}{s}\} + \cos m\alpha\}$$

$$\varphi(n;t,v) = \frac{1}{2}\rho_2^n \{\sin n\beta\, \{\cot \frac{t}{2} - \frac{2}{t}\} + \cos n\beta\}$$

and $\varphi(m;s,u) = O(1)$, $\varphi(n;t,v) = O(1)$, $0 \leq u,v \leq 1$, $0 \leq s,t \leq \pi$, $m,n = 1,2,3,\ldots$; and $\varphi(m;o,u) = \lim_{s \to o} \varphi(m;s,u)$ and $\varphi(n;o,v) = \lim_{t \to o} \varphi(n;t,v)$.

We introduce a lemma due to Livingston [7, p. 311].

(4.5) Lemma

If $1 < r < \exp(8/\pi^2)$, then $\rho_1^2 < r^{-\frac{u}{2}(1-u)s^2}$ uniformly for $0 \leq u \leq 1$ and $0 \leq s \leq \pi$.

Then, also using (2.36),

$$\left|\frac{\rho_1^m}{s}\{\sin m\alpha - \sin msu\}\right|$$

$$< \frac{2r^{-\frac{u}{4}(1-u)ms^2}}{s}\, |\cos\, (msu + O(mu(1-u)s^3))\sin O(mu(1-u)s^3)|$$

$$= O\left(u(1-u)ms^2\, r^{-\frac{u}{4}(1-u)ms^2}\right)$$

$$= o(1)\ ,\quad m \to \infty\ ,$$

uniformly in s and u. Hence we write

$$\rho_1^m \frac{\sin m\alpha}{s} = \rho_1^m \frac{\sin msu}{s} + \psi(m;s,u)$$

$$\rho_2^n \frac{\sin n\beta}{t} = \rho_2^n \frac{\sin ntv}{t} + \psi(n;t,v)$$

where $\psi(m;s,u)$ and $\psi(n;t,v)$ tend uniformly to zero as $m,n \to \infty$. Substituting in (4.4),

$$K_{m,n}(s,t) = \int_{0,0}^{1,1} \{\rho_1^m \frac{\sin msu}{s} + \psi(m;s,u)\}\{\rho_2^n \frac{\sin ntv}{t} + \psi(n;t,v)\} d^2g(u,v)$$

$$+ \int_{0,0}^{1,1} \{\rho_1^m \frac{\sin msu}{s} + \psi(m;s,u)\}\ \varphi(n;t,v) d^2g(u,v)$$

$$+ \int_{0,0}^{1,1} \{\rho_2^n \frac{\sin ntv}{t} + \psi(n;t,v)\}\ \varphi(m;s,u) d^2g(u,v)$$

$$+ \int_{0,0}^{1,1} \varphi(m;s,u)\ \varphi(n;t,v) d^2g(u,v)$$

$$= \int_{0,0}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t}\ d^2g(u,v)$$

$$+ \int_{0,0}^{1,1} \rho_1^m \frac{\sin msu}{s}\ \{\psi(n;t,v) + \varphi(n;t,v)\} d^2g(u,v)$$

$$+ \int_{0,0}^{1,1} \rho_2^n \frac{\sin ntv}{t}\ \{\psi(m;s,u) + \varphi(m;s,u)\} d^2g(u,v)$$

$$+ \int_{0,0}^{1,1} \{\psi(m;s,u) + \varphi(m;s,u)\}\{\psi(n;t,v) + \varphi(n;t,v)\} d^2g(u,v)$$

We denote the mn-th Lebesgue constant for the Hausdorff means relative to the weight function $g(u,v)$ by $L(m,n;g)$. Then, setting

$$\psi(m;s,u) + \varphi(m;s,u) = \gamma(m;s,u)$$
$$\psi(n;t,v) + \varphi(n;t,v) = \gamma(n;t,v) \quad ,$$

where $\gamma(m;s,u) = O(1)$, $\gamma(n;t,v) = O(1)$, $0 \leq u,v \leq 1$, $0 \leq s,t \leq \pi$, we have

$$(4.6) \qquad L(m,n;g) = \frac{4}{\pi^2} \int_{0,0}^{\pi,\pi} |K_{m,n}(s,t)|dsdt$$
$$= \frac{4}{\pi^2} \int_{0,0}^{\pi,\pi} \left| \int_{0,0}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2g(u,v) \right| dsdt$$
$$+ O\left(\int_{0,0}^{\pi,\pi} \left| \int_{0,0}^{1,1} \rho_1^m \frac{\sin msu}{s} \gamma(n;t,v) d^2g(u,v) \right| dsdt \right)$$
$$+ O\left(\int_{0,0}^{\pi,\pi} \left| \int_{0,0}^{1,1} \rho_2^n \frac{\sin ntv}{t} \gamma(m;s,u)\, d^2g(u,v) \right| dsdt \right)$$
$$+ O\left(\int_{0,0}^{\pi,\pi} \left| \int_{0,0}^{1,1} \gamma(m;s,u)\, \gamma(n;t,v) d^2g(u,v) \right| dsdt \right) .$$

Now the last integral is clearly $O(1)$, $m,n = 1,2,3,\ldots$, or $o(\log m)$, $m \to \infty$. To estimate the contribution of the second and third integrals, we prove a lemma.

(4.7) Lemma 1

If $\varphi(m,n;s,t,u,v)$ is uniformly bounded in each argument in the region of integration, and if $g(u,v)$, of bounded variation in the cell $[1,1;0,0]$, is such that its points of discontinuity are distinct from

those of $\varphi(m,n;s,t,u,v)$, then

$$\int_{0,0}^{\pi,\pi} \left| \int_{0,0}^{1,1} \varphi(m,n;s,t,u,v) \frac{\sin msu}{s} d^2g(u,v) \right| dsdt$$

$$= o(\log m \log n) , \quad m \to \infty .$$

Proof

$$\int_{0,0}^{\pi,\pi} \left| \int_{0,0}^{1,1} \varphi(m,n;s,t,u,v) \frac{\sin msu}{s} d^2g(u,v) \right| dsdt$$

$$\leq \int_{0,0}^{\pi,\pi} \int_{0,0}^{1,1} |\varphi(m,n;s,t,u,v)| \left|\frac{\sin msu}{s}\right| |d^2g(u,v)| dsdt$$

$$= O(1) \int_{0,0}^{\pi,\pi} \int_{0,0}^{1,1} \left|\frac{\sin msu}{s}\right| |d^2g(u,v)| dsdt$$

$$= O(1) \int_{0,0}^{\pi,\pi m} \int_{0,0}^{1,1} \left|\frac{\sin su}{s}\right| |d^2g(u,v)| dsdt$$

$$= O(1) \left\{ \int_{0,0}^{\pi,1} + \int_{0,1}^{\pi,\pi m} \right\} \int_{0,0}^{1,1} \left|\frac{\sin su}{s}\right| |d^2g(u,v)| dsdt$$

$$= O(1) + O(1) \int_{1}^{\pi m} \frac{ds}{s}$$

$$= O(\log m) , \quad m \to \infty$$

$$= o(\log m \log n) , \quad n,m \to \infty .$$

We apply the lemma to (4.6) and obtain

(4.8) Theorem 1

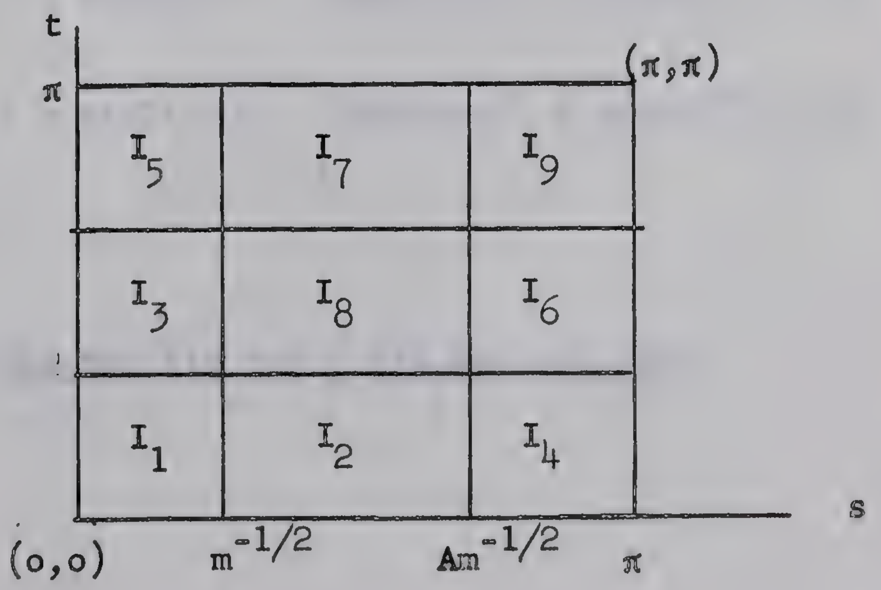
If $L(m,n;g)$ denotes the mn-th Lebesgue contant for the Hausdorff means corresponding to the regular weight function $g(u,v)$, then

$$L(m,n;g) = \frac{4}{\pi^2} \int_{o,o}^{\pi,\pi} \left| \int_{o,o}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2 g(u,v) \right| dsdt$$

$$+ o(\log m \log n) , \quad m,n \to \infty .$$

§ 2.

We wish to obtain an asymptotic estimate for the integral in Theorem 1 as m,n tend to infinity. To this end, we divide the cell $[\pi,\pi;o,o]$ into nine sub-cells and denote the portions of this integral taken over the respective sub-cells by $I_1, I_2, I_3, \ldots, I_9$ as follows, taking $1 < A < \pi m^{1/2}$:



(4.9) Lemma 2

$$I_1 = \int_{o,o}^{n^{-1/2},m^{-1/2}} \left| \int_{o,o}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} \, d^2g(u,v) d^2g(u,v) \right| dsdt$$

$$= \int_{o,o}^{n^{1/2},m^{1/2}} \left| \int_{o,o}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} \, d^2g(u,v) \right| dsdt + o(\log m \log n) \ ,$$

$$m,n \to \infty \ .$$

Proof

Recall (2.28) that

$$\rho_1^2 = 1-4u(1-u) \sin^2 \frac{s}{2} \geq \cos^2 \frac{s}{2} \ .$$

Hence in the cell $[m^{-1/2},n^{-1/2};0,0]$,

$$\rho_1^m \geq \cos^m \frac{s}{2} \geq 1-m(s/2)^2 \geq 3/4$$

$$\rho_2^n \geq \cos^n \frac{t}{2} \geq 1-n(t/2)^2 \geq 3/4 \ .$$

We set

$$(4.10) \qquad \rho_1^m = 1 + \varphi(m;s,u) \ , \quad |\varphi(m;s,u)| \leq m(s/2)^2 \leq 1/4$$

$$\rho_2^n = 1 + \varphi(n;t,v) \ , \quad |\varphi(n;t,v)| \leq n(t/2)^2 \leq 1/4 \ .$$

Then

$$\rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} = \frac{\sin msu}{s} \frac{\sin ntv}{t}$$

$$+ \{\varphi(m;s,u) + \varphi(n;t,v) + \varphi(m;s,u)\varphi(n;t,v)\} \frac{\sin msu}{s} \frac{\sin ntv}{t}$$

and

$$(4.11) \qquad I_1 = \int_{0,0}^{n^{-1/2},m^{-1/2}} \left| \int_{0,0}^{1,1} \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2g(u,v) \right| dsdt$$

$$+ O(1) \int_{0,0}^{n^{-1/2},m^{-1/2}} \left| \int_{0,0}^{1,1} \varphi(m;s,u) \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2g(u,v) \right| dsdt$$

$$+ O(1) \int_{0,0}^{n^{-1/2},m^{-1/2}} \left| \int_{0,0}^{1,1} \varphi(n;t,v) \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2g(u,v) \right| dsdt$$

$$+ O(1) \int_{0,0}^{n^{-1/2},m^{-1/2}} \left| \int_{0,0}^{1,1} \varphi(m;s,u)\varphi(n;t,v) \right.$$

$$\left. \cdot \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2g(u,v) \right| dsdt \quad .$$

Taking (4.10) into account, $\left|\varphi(m;s,u) \frac{\sin msu}{s}\right| \leq \frac{ms}{4}$, and so

$$\int_{0,0}^{n^{-1/2},m^{-1/2}} \left| \int_{0,0}^{1,1} \varphi(m;s,u) \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2g(u,v) \right| dsdt$$

$$\leq \int_{0,0}^{n^{-1/2},m^{-1/2}} ms \int_{0,0}^{1,1} \left|\frac{\sin ntv}{t}\right| |d^2g(u,v)| dsdt$$

$$= \frac{1}{2} \int_{0}^{n^{1/2}} \int_{0,0}^{1,1} \left|\frac{\sin ntv}{t}\right| |d^2g(u,v)| dt$$

$$= O(1) + \frac{1}{2} \int_{0,0}^{1,1} \int_{1}^{n^{1/2}} \left|\frac{\sin tv}{t}\right| dt \, |d^2g(u,v)|$$

$$= O(1) + O(1)\log n$$

$$= o(\log m \log n)\ , \quad m,n \to \infty\ .$$

Hence the contribution of the second and also of the third term in (4.11) is $o(\log m \log n)$ as $m,n \to \infty$. The contribution of the fourth term is bounded by

$$\int_{o,o}^{n^{-1/2},m^{-1/2}} msnt \int_{o,o}^{1,1} |d^2g(u,v)|dsdt$$

$$= \frac{1}{4} V(g) = o(\log m \log n)\ , \quad m,n \to \infty\ .$$

Hence

(4.12) $$I_1 = \int_{o,o}^{n^{-1/2},m^{-1/2}} \left| \int_{o,o}^{1,1} \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2g(u,v) \right| dsdt + o(\log m \log n)$$

$$= \int^{n^{1/2},m^{1/2}} \left| \int_{o,o}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} d^2g(u,v) \right| dsdt$$

$$+ o(\log m \log n)\ , \quad m,n \to \infty\ .$$

Next, shifting the lower limit of integration with respect to s and t from o to 1 in (4.12) introduces an error of

$$\left\{ \int_{o,o}^{1,1} + \int_{o,1}^{1,m^{1/2}} + \int_{1,o}^{n^{1/2},1} \right\} \left| \int_{o,o}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} d^2g(u,v) \right| dsdt$$

The first integral is clearly $O(1)$. The second and third integrals

are $o(\log m \log n)$, $m,n \to \infty$. Hence

$$I_1 = \int_{1,1}^{n^{1/2},m^{1/2}} \left| \int_{0,0}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} \, d^2 g(u,v) \right| dsdt$$

$$+ \, o(\log m \log n) \, , \quad m,n \to \infty \, .$$

This proves Lemma 2.

(4.13) Lemma 3

$$I_2 = \int_{0,m^{-1/2}}^{n^{-1/2},Am^{-1/2}} \left| \int_{0,0}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} \, d^2 g(u,v) \right| dsdt$$

$$= o(\log m \log n) \, , \quad m,n \to \infty \, .$$

$$I_3 = \int_{n^{-1/2},\,0}^{An^{-1/2},m^{-1/2}} \left| \int_{0,0}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} \, d^2 g(u,v) \right| dsdt$$

$$= o(\log m \log n) \, , \quad m,n \to \infty \, .$$

Proof.

$$I_2 = \int_{0,m^{-1/2}}^{n^{-1/2},Am^{-1/2}} \left| \int_{0,0}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} \, d^2 g(u,v) \right| dsdt$$

$$\leq \int_{0,m^{1/2}}^{n^{1/2},Am^{1/2}} \left| \int_{0,0}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} \, d^2 g(u,v) \right| dsdt$$

$$\leq \left\{ \int_0^1 + \int_1^{n^{1/2}} \right\} \int_{m^{1/2}}^{Am^{1/2}} \int_{0,0}^{1,1} \left|\frac{\sin su}{s}\right| \left|\frac{\sin tv}{t}\right| |d^2 g(u,v)|$$

$$\leq V(g) \log A \; \{1 + \tfrac{1}{2} \log n\}$$

$$= o(\log m \log n) , \quad m,n \to \infty \; .$$

This proves half of the lemma. The other half follows by a parallel argument.

(4.14) Lemma 4

$$I_4 = \int_{0,Am^{-1/2}}^{n^{-1/2},\pi} \left| \int_{0,0}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2 g(u,v) \right| ds dt$$

$$= \frac{1}{\pi} \log m \int_1^{n^{1/2}} \left| \int_0^1 \frac{\sin tv}{t} \{dg(1,v) - dg(1^-,v)\} \right| dt$$

$$= o(\log m \log n) , \quad m,n \to \infty \; .$$

$$I_5 = \int_{An^{-1/2},0}^{\pi,m^{-1/2}} \left| \int_{0,0}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2 g(u,v) \right| ds dt$$

$$= \frac{1}{\pi} \log n \int_1^{n^{1/2}} \left| \int_0^1 \frac{\sin su}{s} \{dg(u,1) - dg(u,1^-)\} \right| ds$$

$$= o(\log m \log n) , \quad m,n \to \infty \; .$$

Proof

Again, we will prove the first half of the lemma, the proof of the second half being essentially the same. We introduce a lemma due to Lorch and Newman [8, p. 287].

(4.15) Lemma

For $Am^{-1/2} \leq s \leq \pi, \quad \rho_1^m < e^{-2(2A/\pi)^2 u(1-u)} = \psi(A,u).$

We write

$$(4.16) \qquad I_4 = \int_{0,Am^{-1/2}}^{n^{-1/2},\pi} \left| \int_{0,0}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} \, d^2g(u,v) \right| dsdt$$

$$= \int_{0,Am^{-1/2}}^{n^{-1/2},\pi} \left| \int_{0,0}^{1^-,1^-} + \int_{0,1^-}^{1,1} + \int_{1^-,0}^{1,1^-} \right| dsdt$$

$$= \int_{0,Am^{-1/2}}^{n^{-1/2},\pi} \left| \int_{0,1^-}^{1,1} \right| dsdt + O(1) \int_{0,Am^{-1/2}}^{n^{-1/2},\pi} \left| \int_{0,0}^{1^-,1^-} \right| dsdt$$

$$+ O(1) \int_{0,Am^{-1/2}}^{n^{-1/2},\pi} \left| \int_{1^-,0}^{1,1^-} \right| dsdt \quad .$$

Now

$$(4.17) \qquad \int_{0,Am^{-1/2}}^{n^{-1/2},\pi} \left| \int_{1^-,0}^{1,1^-} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} \, d^2g(u,v) \right| dsdt$$

$$= \int_{o,Am^{-1/2}}^{n^{-1/2},\pi} \left| \int_{1^-,o}^{1,1^-} \rho_1^m \frac{\sin msu}{s} \frac{\sin nt}{t} d^2g(u,v) \right| dsdt$$

$$\leq \int_o^{n^{1/2}} \left|\frac{\sin t}{t}\right| dt \int_{Am^{-1/2}}^{\pi} \int_{1^-,o}^{1,1^-} \psi(A,u) \left|\frac{\sin msu}{s}\right| |d^2g(u,v)| ds$$

$$\leq \{1 + \tfrac{1}{2} \log n\} \int_{Am^{1/2}}^{\pi m} \frac{ds}{s} \int_{1^-,o}^{1,1^-} \psi(A,u) \; |d^2g(u,v)|$$

$$= \{\tfrac{1}{2} \log n + 1\} \{\tfrac{1}{2} \log m + \log \tfrac{\pi}{A}\} \int_{1^-,o}^{1,1^-} \psi(A,u) \; |d^2g(u,v)|$$

$$= \frac{1}{4} \log m \log n \int_{1^-,o}^{1,1^-} \psi(A,u) \; |d^2g(u,v)| + o(\log m \log n) \; , \quad m,n \to \infty \; .$$

Next,

$$(4.18) \qquad \int_{o,Am^{-1/2}}^{n^{-1/2},\pi} \left| \int_{o,o}^{1^-,1^-} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2g(u,v) \right| dsdt$$

$$\leq \int_{o,Am^{-1/2}}^{n^{-1/2},\pi} \int_{o,o}^{1^-,1^-} \psi(A,u) \left|\frac{\sin msu}{s} \frac{\sin ntv}{t}\right| |d^2g(u,v)| dsdt$$

$$= \left\{ \int_o^1 + \int_1^{n^{1/2}} \right\} \int_{Am^{1/2}}^{\pi m} \int_{o,o}^{1,1} \psi(A,u) \left|\frac{\sin su}{s}\right| \left|\frac{\sin tv}{t}\right| |d^2g(u,v)| dsdt$$

$$\leq \left\{1 + \int_1^{n^{1/2}} \frac{dt}{t}\right\} \int_{Am^{1/2}}^{\pi m} \frac{ds}{s} \int_{0,0}^{1^-,1^-} \psi(A,u)\,|d^2g(u,v)|$$

$$= \{1 + \tfrac{1}{2}\log n\}\{\tfrac{1}{2}\log m + \log \tfrac{\pi}{A}\} \int_{0,0}^{1^-,1^-} \psi(A,u)\,|d^2g(u,v)|$$

$$= \frac{1}{4}\log m \log n \int_{0,0}^{1^-,1^-} \psi(A,u)\,|d^2g(u,v)| + O(\log m \log n)\ , \quad m,n \to \infty\ .$$

Finally,

$$(4.19) \qquad \int_{0,Am^{-1/2}}^{n^{-1/2},\pi} \left| \int_{0,1^-}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t}\, d^2g(u,v) \right| dsdt$$

$$= \int_{0,Am^{-1/2}}^{n^{-1/2},\pi} \left| \int_0^1 \rho_2^n \frac{\sin ntv}{t} \frac{\sin ms}{s} \{dg(1,v) - dg(1^-,v)\} \right| dsdt$$

$$= \int_{0,Am^{-1/2}}^{n^{-1/2},\pi} \left| \int_0^1 \frac{\sin ntv}{t} \frac{\sin ms}{s} \{dg(1,v) - dg(1^-,v)\} \right| dsdt$$

$$+ O(1) \int_{0,Am^{-1/2}}^{n^{-1/2},\pi} \left| \int_0^1 \varphi(n;t,v) \frac{\sin ntv}{t} \frac{\sin ms}{s} \{dg(1,v) - dg(1^-,v)\} \right| dsdt .$$

Now by (4.10), $\varphi(n;t,v)\sin ntv = O(nt^2)$. Hence the last term above is dominated by

$$O(1) \int_0^{n^{-1/2}} O(nt)dt \int_{Am^{-1/2}}^{\pi} \frac{ds}{s} \int_0^1 |dg(1,v) - dg(1^-,v)|$$

$$= O(\log m)$$

$$= o(\log m \log n) \ , \ n, m \to \infty \quad .$$

As for the first term in (4.19), we change the variables s and t and get

$$\int_{0,Am^{-1/2}}^{n^{-1/2},\pi} \left| \int_0^1 \frac{\sin ms}{s} \frac{\sin ntv}{t} \{dg(1,v) - dg(1^-,v) \right| dsdt$$

$$= \int_0^{n^{1/2}} \int_{Am^{1/2}}^{\pi m} \left|\frac{\sin s}{s}\right| ds \left| \int_0^1 \frac{\sin tv}{t} \{dg(1,v) - dg(1^-,v)\} \right| dt$$

$$= \int_{Am^{1/2}}^{\pi m} \frac{|\sin s|}{s} ds \left\{ \int_0^1 + \int_1^{n^{1/2}} \right\} \left| \int_0^1 \frac{\sin tv}{t} \{dg(1,v) - dg(1^-,v)\} \right| dt$$

$$= \int_{Am^{1/2}}^{\pi m} \frac{|\sin s|}{s} ds \int_0^1 \left| \int_0^1 \frac{\sin tv}{t} \{dg(1,v) - dg(1^-,v)\} \right| dt$$

$$+ \int_{Am^{1/2}}^{\pi m} \frac{|\sin s| - \frac{2}{\pi}}{s} ds \int_1^{n^{1/2}} \left| \int_0^1 \frac{\sin tv}{t} \{dg(1,v) - dg(1^-,v)\} \right| dt$$

$$+ \frac{2}{\pi} \int_{Am^{1/2}}^{\pi m} \frac{ds}{s} \int_1^{n^{1/2}} \left| \int_0^1 \frac{\sin tv}{t} \{dg(1,v) - dg(1^-,v)\} \right| dt$$

$$= O(\log m) + o(\log n)$$

$$+ \frac{2}{\pi} \{\log \pi m - \log Am^{1/2}\} \int_1^{n^{1/2}} \left| \int_0^1 \frac{\sin tv}{t} \{dg(1,v) - dg(1^-,v)\} \right| dt$$

$$= \{\frac{1}{\pi} \log m + \frac{2}{\pi} \log \frac{\pi}{A}\} \int_1^{n^{1/2}} \left| \int_0^1 \frac{\sin tv}{t} \{dg(1,v) - dg(1^-,v)\} \right| dt$$

$$+ O(\log m) + o(\log n)$$

$$= \frac{1}{\pi} \log m \int_1^{n^{1/2}} \left| \int_0^1 \frac{\sin tv}{t} \{dg(1,v) - dg(1^-,v)\} \right| dt$$

$$+ O(\log m) + O(\log n)$$

$$= \frac{1}{\pi} \log m \int_1^{n^{1/2}} \left| \int_0^1 \frac{\sin tv}{t} \{dg(1,v) - dg(1^-,v)\} \right| dt$$

$$+ o(\log m \log n) , \quad m,n \to \infty .$$

Then by (4.16), (4.17), (4.18) and (4.19),

$$I_4 = \frac{1}{\pi} \log m \int_1^{n^{1/2}} \left| \int_0^1 \frac{\sin tv}{t} \{dg(1,v) - dg(1^-,v)\} \right| dt$$

$$+ O(1) \log m \log n \int_{0,0}^{1^-,1^-} \psi(A,u) |d^2 g(u,v)|$$

$$+ O(1) \log m \log n \int_{1^-,0}^{1,1^-} \psi(A,u) |d^2 g(u,v)|$$

$$+ o(\log m \log n) \ , \quad n,m \to \infty \ .$$

Now the integrals

$$\int_{o,o}^{1^-,1^-} \psi(A,u)\,|d^2g(u,v)| \ , \quad \int_{1^-,o}^{1,1^-} \psi(A,u)\,|d^2g(u,v)|$$

tend to zero as $A \to \infty$ by the dominated convergence theorem, since $g(o,v) = g(o^+,v)$, and $\psi(A,u) \to o$ boundedly on $(0,1)$. Hence letting $A \to \infty$ as $m,n \to \infty$ completes the proof of the first half of the lemma. As mentioned earlier, the second half follows by a parallel proof.

(4.20) Lemma 5

$$I_6 = \int_{n^{-1/2},\,Am^{-1/2}}^{Am^{-1/2},\,\pi} \left| \int_{o,o}^{1,1} \rho_1^m \rho_2^n \,\frac{\sin msu}{s}\, \frac{\sin ntv}{t}\, d^2g(u,v) \right| dsdt$$

$$= o(\log m \log n) \ , \quad m,n \to \infty$$

$$I_7 = \int_{An^{-1/2},\,m^{-1/2}}^{\pi,\,Am^{-1/2}} \left| \int_{o,o}^{1,1} \rho_1^m \rho_2^n \,\frac{\sin msu}{s}\, \frac{\sin ntv}{t}\, d^2g(u,v) \right| dsdt$$

$$= o(\log m \log n) \ , \quad m,n \to \infty \ .$$

Proof

Again we prove the first half of the lemma, the second half following by a parallel argument.

$$I_6 = \int_{n^{-1/2},\,Am^{-1/2}}^{An^{-1/2},\,\pi} \left| \int_{o,o}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2g(u,v) \right| dsdt$$

$$\leq \int_{n^{-1/2}}^{An^{-1/2}} \frac{dt}{t} \int_{Am^{-1/2}}^{\pi} \frac{ds}{s} \cdot V(g)$$

$$= V(g) \log A \log \{(\pi/A)m^{1/2}\}$$

$$= o(\log m \log n) , \quad m,n \to \infty .$$

(4.21) Lemma 6

$$I_8 = \int_{n^{-1/2},\,m^{-1/2}}^{An^{-1/2},\,Am^{-1/2}} \left| \int_{o,o}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2g(u,v) \right| dsdt$$

$$= o(\log m \log n) , \quad m,n \to \infty$$

Proof

$$I_8 = \int_{n^{-1/2},\,m^{-1/2}}^{An^{-1/2},\,Am^{-1/2}} \left| \int_{o,o}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2g(u,v) \right| dsdt$$

$$\leq \int_{n^{-1/2}}^{An^{-1/2}} \frac{dt}{t} \int_{m^{-1\ 2}}^{Am^{-1/2}} \frac{ds}{s} \cdot V(g)$$

$$= V(g)(\log A)^2 = o(\log m \log n) , \quad m,n \to \infty .$$

(4.22) Lemma 7

$$I_9 = \int_{An^{-1/2},Am^{-1/2}}^{\pi,\pi} | \int_{o,o}^{1,1} \rho_1^m\rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2g(u,v) | dsdt$$

$$= \frac{1}{\pi^2} g(1,1;1^-,1^-) \log m \log n + o(\log m \log n) , \quad m,n \to \infty .$$

Proof

$$I_9 = \int_{An^{-1/2},Am^{-1/2}}^{\pi,\pi} | \int_{o,o}^{1,1} \rho_1^m\rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2g(u,v) | dsdt$$

$$= \int_{An^{-1/2},Am^{-1/2}}^{\pi,\pi} | \int_{1^-,1^-}^{1,1} | dsdt + O(1) \int_{An^{-1/2},Am^{-1/2}}^{\pi,\pi} | \int_{o,o}^{1,1^-} | dsdt$$

$$+ O(1) \int_{An^{-1/2},Am^{-1/2}}^{\pi,\pi} | \int_{o,1^-}^{1^-,1} | dsdt .$$

Now

$$(4.23) \quad \int_{An^{-1/2},Am^{-1/2}}^{\pi,\pi} | \int_{o,1^-}^{1^-,1} \rho_1^m\rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2g(u,v) | dsdt$$

$$= \int_{An^{-1/2},Am^{-1/2}}^{\pi,\pi} \rho_2^n \frac{\sin ms}{s} \frac{\sin ntv}{t} d^2g(u,v) | dsdt$$

$$\leq \int_{An^{1/2},Am^{1/2}}^{\pi n,\pi m} \frac{ds}{s} \frac{dt}{t} \int_{o,1}^{1^-,1} \psi(A,v) | d^2g(u,v) |$$

$$= \frac{1}{4} \log m \log n \int_{0,1^-}^{1^-,1} \psi(A,v) |d^2 g(u,v)| + o(\log m \log n) , \quad m,n \to \infty .$$

$$(4.24) \qquad \int_{An^{-1/2},Am^{-1\ 2}}^{\pi,\pi} \left| \int_{0,0}^{1,1^-} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2 g(u,v) \right| dsdt$$

$$\leq \int_{An^{1/2},Am^{1/2}}^{\pi n,\pi m} \frac{ds}{s} \frac{dt}{t} \int_{0,0}^{1,1^-} \psi(A,u) |d^2 g(u,v)|$$

$$= \frac{1}{4} \log m \log n \int_{0,0}^{1,1^-} \psi(A,u) |d^2 g(u,v)| + o(\log m \log n) , \quad m,n \to \infty .$$

$$(4.25) \qquad \int_{An^{-1/2},Am^{-1/2}}^{\pi,\pi} \left| \int_{1^-,1^-}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} d^2 g(u,v) \right| dsdt$$

$$= \int_{An^{-1/2},Am^{-1/2}}^{\pi,\pi} \left| \frac{\sin ms}{s} \frac{\sin nt}{t} g(1,1;1^-,1^-) \right| dsdt$$

$$= \int_{An^{1/2}}^{\pi n} \frac{|\sin t|}{t} ds \int_{Am^{1/2}}^{\pi m} \frac{|\sin s|}{s} dt \; |g(1,1;1^-,1^-)| \quad .$$

Now

$$\int_{Am^{1/2}}^{\pi m} \frac{|\sin s|}{s} ds = \int_{An^{1/2}}^{\pi n} \frac{|\sin s| - \frac{2}{\pi} + \frac{2}{\pi}}{s} ds$$

$$= \frac{2}{\pi} \int_{Am^{1/2}}^{\pi m} \frac{ds}{s} + \int_{Am^{1/2}}^{\pi m} \frac{|\sin s| - \frac{2}{\pi}}{s} ds$$

$$= \frac{2}{\pi}\left\{\frac{1}{2}\log m + \log(\pi/A)\right\} + O(1)$$

$$= \frac{1}{\pi}\log m + O(1)\,, \qquad m \to \infty \;.$$

Hence

$$\int_{An^{-1/2},\,Am^{-1/2}}^{\pi,\pi} \left| \int_{1^-,1^-}^{1,1} \rho_1^m \rho_2^n \,\frac{\sin msu}{s}\,\frac{\sin ntv}{t}\, d^2 g(u,v)\right| ds\,dt$$

$$= \frac{1}{\pi^2}\log m \log n \,|g(1,1;1^-,1^-)| + o(\log m \log n)\,, \quad m,n \to \infty \;.$$

Collecting the results, we obtain

$$I_9 = \frac{1}{\pi^2}\log m \log n \,|g(1,1;1^-,1^-)|$$

$$+\; O(1) \log m \log n \int_{0,0}^{1^-,1^-} \psi(A,u)\,|d^2 g(u,v)|$$

$$+\; O(1) \log m \log n \int_{0,1^-}^{1^-,1} \psi(A,v)\,|d^2 g(u,v)|$$

$$+\; o(\log m \log n)\,, \quad m,n \to \infty \;.$$

$$= \frac{1}{\pi^2}\log m \log n \,|g(1,1;1^-,1^-)| + o(\log m \log n), \quad m,n \to \infty \;,$$

where the last equality follows the same argument as used in the proof

of Lemma 4. This completes the proof of Lemma 7.

We collect the results of Lemmas 2 to 7, and state the result in a theorem

(4.26) Theorem 2

If $L(m,n;g)$ denotes the mn-th Lebesgue constant for the Hausdorff means corresponding to the regular weight function $g(u,v)$, then

$$L(m,n;g) = \frac{4}{\pi^2} \int_{1,1}^{n^{1/2},m^{1/2}} \left| \int_{o,o}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} \, d^2 g(u,v) \right| ds dt$$

$$+ \frac{4}{\pi^3} \log n \int_1^{m^{1/2}} \left| \int_o^1 \frac{\sin su}{s} \{dg(u,1) - dg(u,1^-)\} \right| ds$$

$$+ \frac{4}{\pi^3} \log m \int_1^{n^{1/2}} \left| \int_o^1 \frac{\sin tv}{t} \{dg(1,v) - dg(1^-,v)\} \right| dt$$

$$+ \frac{4}{\pi^4} |g(1,1;1^-,1^-)| \log m \log n + o(\log m \log n) , \quad m,n \to \infty .$$

(4.27) Remarks

If $g(u,1) = g(u,1^-)$ and $g(1,v) = g(1^-,v)$, $o \le u,v < 1$, then

$$L(m,n;g) = \frac{4}{\pi^2} \int_{1,1}^{n^{1/2},m^{1/2}} \left| \int_{o,o}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} \, d^2 g(u,v) \right| ds dt$$

$$+ \frac{4}{\pi^4} \; |g(1,1;1^-,1^-)| \log m \log n + o(\log m \log n) \; , \quad m,n \to \infty \; .$$

If, in addition, $g(u,v)$ is continuous at $(1,1)$, or if only $g(1,1;1^-,1^-) = 0$, then

$$L(m,n;g) = \frac{4}{\pi^2} \int_{1,1}^{n^{1/2},m^{1/2}} \left| \int_{o,o}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} \, d^2 g(u,v) \right| ds dt$$

$$+ o(\log m \log n) \; , \quad m,n \to \infty \; .$$

§ 3.

Let $\{u_i, v_j\}$ be an enumeration of all the mass points of the function $g(u,v)$ in the cell $[1,1;o,o]$, and define a function $g_1(u,v)$ by

$$g_1(u,v) = \sum_{u_i \le u, v_j \le v} g(u_i^+, v_j^+; u_i^-, v_j^-) \; .$$

Then

$$g_2(u,v) = g(u,v) - g_1(u,v)$$

is a function of bounded variation which has no mass points. Then

$$\int_{o,o}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} \, d^2 g(u,v)$$

$$= \int_{o,o}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} \, d^2 g_1(u,v) + \int_{o,o}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} \, d^2 g_2(u,v)$$

$$= \frac{1}{st} \sum_{i,j} \sin su_i \sin tv_j \; g(u_i^+, v_j^+; u_i^-, v_j^-)$$

$$+ \int_{o,o}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} \, d^2 g_2(u,v)$$

and

$$\int_{1,1}^{n^{1/2},m^{1/2}} \left| \int_{o,o}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} \, d^2 g(u,v) \right| dsdt$$

$$= \int_{1,1}^{n^{1/2},m^{1/2}} \frac{1}{st} \left| \sum_{i,j} \sin su_i \sin tv_j \; g(u_i^+, v_j^+; u_i^-, v_j^-) \right| dsdt$$

$$+ O(1) \int_{1,1}^{n^{1/2},m^{1/2}} \left| \int_{o,o}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} \, d^2 g_2(u,v) \right| dsdt$$

$$= \int_{1,1}^{n^{1/2},m^{1/2}} \frac{1}{st} \left| \sum_{i,j} \sin su_i \sin tv_j \; g(u_i^+, v_j^+; u_i^-, v_j^-) \right| dsdt$$

$$+ o(\log m \log n) , \quad m,n \to \infty ,$$

where the last result follows by an application of Theorem 6 of Chapter 3.

We write

$$f(s,t) = \Big| \sum_{i,j} \sin su_i \sin tv_j \; g(u_i^+,v_j^+;u_i^-,v_j^-) \Big|$$

and define the mean value of $f(s,t)$ by

$$M\{f(s,t)\} = \lim_{p,q \to \infty} \frac{1}{pq} \int_{o,o}^{p,q} f(s,t)dsdt \quad . \tag{4.28}$$

To prove that the mean value exists, set

$$f_{k,\ell}(s,t) = \Big| \sum_{i,j=1}^{k,\ell} \sin su_i \sin tv_j \; g(u_i^+,v_j^+;u_i^-,v_j^-) \Big|$$

and

$$f(s,t) = f_{k,\ell}(s,t) + \epsilon_{k,\ell}(s,t)$$

$$\epsilon_{k,\ell} = \sup_{s,t} |\epsilon_{k,\ell}(s,t)| \quad .$$

Since $g(u,v)$ is of bounded variation, $f(s,t)$ exists for all s,t, and in fact, $o \leq f(s,t) \leq V(g)$. The sum

$$\sum_{i,j} \sin su_i \sin tv_j \; g(u_i^+,v_j^+;u_i^-,v_j^-)$$

converges absolutely and uniformly in s,t, so that $\epsilon_{k,\ell} \to 0$ as $k,\ell \to \infty$. Then

$$f_{k,\ell}(s,t) - \epsilon_{k,\ell} \le f(s,t) \le f_{k,\ell}(s,t) + \epsilon_{k,\ell}$$

so that

$$\lim_{p,q \to \infty} \frac{1}{pq} \int_{0,0}^{p,q} f_{k,\ell}(s,t)dsdt - \epsilon_{k,\ell} \le \lim_{p,q \to \infty} \frac{1}{pq} \int_{0,0}^{p,q} f(s,t)dsdt$$

$$\le \lim_{p,q \to \infty} \frac{1}{pq} \int_{0,0}^{p,q} f_{k,\ell}(s,t)dsdt + \epsilon_{k\ell} .$$

Now for each k,ℓ, $f_{k,\ell}(s,t)$ is a finite sum of periodic functions, and so its mean value,

$$M\{f_{k,\ell}(s,t)\} = \lim_{p,q \to \infty} \frac{1}{pq} \int_{0,0}^{p,q} f_{k,\ell}(s,t)dsdt$$

exists. Hence

$$\left| \lim_{p,q \to \infty} \frac{1}{pq} \int_{0,0}^{p,q} f(s,t)dsdt - M\{f_{k,\ell}(s,t)\} \right| \le \epsilon_{k,\ell} ,$$

so that $\{M\{f_{k,\ell}(s,t)\}\}$ is a Cauchy sequence and converges to

$$M\{f(s,t)\} = \lim_{p,q \to \infty} \frac{1}{pq} \int_{0,0}^{p,q} f(s,t)dsdt$$

as $k,\ell \to \infty$. This proves the existence of $M\{f(s,t)\}$.

By (4.28), we write

$$\int_{0,0}^{q,p} f(s,t)dsdt = pq\, M\{f(s,t)\} + o(pq) \ , \quad p,q \to \infty \ .$$

Let

$$h(p,q) = \int_{0,0}^{q,p} f(s,t)dsdt$$

Then

$$\frac{\partial^2 h}{\partial p \partial q} = f(p,q) = M\{f(s,t)\} + o(1) \ , \quad p,q \to \infty$$

and

$$(4.29) \quad \int_{1,1}^{n^{1/2},m^{1/2}} \frac{1}{pq}\, f(p,q)dpdq = \int_{1,1}^{n^{1/2},m^{1/2}} \frac{1}{pq}\, \{M\{f(s,t)\} + o(1)\}dpdq$$

$$= \frac{1}{4} M\{f(s,t)\} \log m \log n + o(\log m \log n) \ , \quad m,n \to \infty \ .$$

We examine the second and third integrals in Theorem 2 in the same manner and let $\{u_i\}$ and $\{v_j\}$ be an enumeration of the points $(u_i,1)$, $(1,v_j)$ at which the difference functions $g(u,1) - g(u,1^-)$, resp. $g(1,v) - g(1^-,v)$, are discontinuous. We write

$$g(u,1) - g(u,1^-) = h_1(u) + j_1(u)$$

$$g(1,v) - g(1^-,v) = h_2(v) + j_2(v) \quad ,$$

where $h_1(u)$ and $h_2(v)$ are continuous parts of the functions on the left, and

$$j_1(u) = \sum_{u_i \le u} g(u_i^+,1;u_i^-,1^-)$$

$$j_2(v) = \sum_{v_j \le v} g(1,v_j^+;1^-,v_j^-) \quad .$$

Then

$$\int_1^{m^{1/2}} \left| \int_0^1 \frac{\sin su}{s} \{dg(u,1) - dg(u,1^-)\} \right| ds$$

$$= \int_1^{m^{1/2}} \left| \int_0^1 \frac{\sin su}{s} \{dh_1(u) + dj_1(u)\} \right| ds$$

$$= \int_1^{m^{1/2}} \left| \int_0^1 \frac{\sin su}{s} dj_1(u) \right| ds + O(1) \int_1^{m^{1/2}} \left| \int_0^1 \frac{\sin su}{s} dh_1(u) \right| ds$$

$$= \int_1^{m^{1/2}} \frac{1}{s} \left| \sum_i \sin su_i \, g(u^+,1;u^-,1^-) \right| ds + o(\log m), \quad m \to \infty$$

$$= \int_1^{m^{1/2}} \frac{f_1(s)}{s}\, ds + o(\log m) \ , \quad m \to \infty \ ,$$

where we have used the fact that $h_1(u)$ is continuous and applied (3.1), and then used the obvious substition. Finally, following an argument of Lorch and Newman [8, p. 290],

$$\int_0^p f_1(s) ds = pM\{f_1(s)\} + o(p) \ , \quad p \to \infty \ ,$$

so that

$$f_1(p) = \frac{d}{dp} \int_0^p f_1(s) ds = M\{f_1(s)\} + o(1) \ , \quad p \to \infty$$

and

$$\int_1^{m^{1/2}} \frac{f_1(p)}{p}\, dp = \int_1^{m^{1/2}} \{M\{f_1(s)\} + o(1)\} \frac{dp}{p}$$

$$= \frac{1}{2} M\{f_1(s)\} + o(\log m) \ , \quad m \to \infty \ .$$

Hence

$$(4.30) \quad \int_1^{m^{1/2}} \left| \int_0^1 \frac{\sin su}{s} \{dg(u,1) - dg(u,1^-)\} \right| ds$$

$$= \frac{1}{2} M\{f_1(s)\} \log m + o(\log m) \ , \quad m \to \infty \ .$$

In a similar manner, we obtain

(4.31) $$\int_1^{n^{1/2}} \left| \int_o^1 \frac{\sin tv}{t} \{dg(1,v) - dg(1^-,v)\} \right| dt$$

$$= \frac{1}{2} M\{f_2(t)\} \log n + o(\log n) , \quad n \to \infty ,$$

where now $M\{f_2(t)\}$ is the mean value of

$$f_2(t) = \left| \sum_j \sin tv_j \; g(1,v^+;1^-,v^-) \right| .$$

Then, substituting (4.29), (4.30) and (4.31) in Theorem 2, we have

(4.32) $$L(m,n;g) = \frac{1}{\pi^2} \{M\{f(s,t)\} + \frac{2}{\pi} M\{f_1(s)\} + \frac{2}{\pi} M\{f_2(t)\}$$

$$+ \frac{4}{\pi^2} |g(1,1;1^-,1^-)|\} \log m \log n + o(\log m \log n), \quad m,n \to \infty .$$

We state the result in a theorem.

(4.33) Theorem 3

Let $g(u,v)$ be a weight function corresponding to a regular Hausdorff summability method for double sequences. Let $\{u_i,v_j\}$, $\{u_i\}$ and $\{v_j\}$ be, respectively, an enumeration of the mass points of $g(u,v)$ in the cells $[1,1;o,o]$, $[1,1;o,1^-)$ and $[1,1;1^-,o)$. Let $M\{f\}$ represent the mean value of the function f , and let

$$f(s,t) = \left| \sum_{i,j} \sin su_i \sin tv_j \; g(u_i^+,v_j^+;u_i^-,v_j^-) \right|$$

$$f_1(s) = \left| \sum_i \sin su_i \; g(u_i^+,1;u_i^-,1^-) \right|$$

$$f_2(t) = \left| \sum_j \sin tv_j \; g(1,v_j^+;1^-,v_j^-) \right|$$

The Lebesgue constants, $L(m,n;g)$, for the Hausdorff means corresponding to the weight function $g(u,v)$ are given by

$$L(m,n;g) = \frac{1}{\pi^2} \{M\{f(s,t)\} + \frac{2}{\pi} M\{f_1(s)\} + \frac{2}{\pi} M\{f_2(t)\}$$

$$+ \frac{4}{\pi^2} |g(1,1;1^-,1^-)|\} \log m \log n + o(\log m \log n), \quad m,n \to \infty .$$

§ 4.

By Theorem 3, it follows immediately that if $g(u,v)$ has no mass points, then

(4.34) $$L(m,n;g) = o(\log m \log n) , \quad m,n \to \infty .$$

(4.35) Remarks

We note, in particular, that (4.34) holds for weight functions which may be discontinuous, provided that they have no mass points. This would be the class of weight functions for which the modulus of continuity,

$\omega(g;\delta_1\delta_2)$, defined in (3.21), tends to zero as $\delta_1,\delta_2 \to 0$.

That the estimate (4.34) cannot be improved even for the class of absolutely continuous weight functions follows as an immediate corollary to the following theorem due to Lorch and Newman [10, p. 284].

(4.36) Theorem

Let $\epsilon(m) \to 0$ as $m \to \infty$, $\epsilon(m) > 0$. There exists an increasing, absolutely continuous weight function $g(u)$ for which

$$L(m;g) \neq o(\epsilon(m) \log m) , \qquad m \to \infty .$$

For let $g_1(u)$ and $g_2(v)$ be two such functions. Then for $g(u,v) = g_1(u) \cdot g_2(v)$,

$$\begin{aligned} L(m,n;g) &= L(m;g_1) \cdot L(n;g_2) \\ &\neq o(\delta(m,n) \log m \log n) , \qquad m,n \to \infty , \end{aligned}$$

where $\delta(m,n) = \epsilon(m)\ \epsilon(n) \to 0$ as $m,n \to \infty$.

Going back to Theorem 3,

$$\begin{aligned} f(s,t) &= \Big| \sum_{i,j} \sin su_i \sin tv_j \ g(u_i^+,v_j^+;u_i^-,v_j^-) \Big| \\ &\leq \sum_{i,j} |\sin su_i \sin tv_j| \, |g(u_i^+,v_j^+;u_i^-,v_j^-)| \end{aligned}$$

Hence

$$M\{f(s,t)\} \leq \sum_{i,j} M\{|\sin su_i \sin tv_j|\}|g(u_i^+,v_j^+;u_i^-,v_j^-)|$$

$$= \frac{4}{\pi^2} \sum_{i,j} |g(u_i^+,v_j^+;u_i^-,v_j^-)|$$

$$\leq \frac{4}{\pi^2} V(g) .$$

Similarly,

$$M\{f_1(s)\} \leq \frac{2}{\pi} V(g)$$

$$M\{f_2(t)\} \leq \frac{2}{\pi} V(g) .$$

Hence by Theorem 3, taking (4.34) into account, and noting that also $|g(1,1;1^-,1^-)| \leq V(g)$, we may write

$$L(m,n;g) = c(g) \log m \log n + o(\log m \log n) , \quad m,n \to \infty , \tag{4.37}$$

where

$$c(g) = \frac{1}{\pi^2} \{M\{f(s,t)\} + \frac{2}{\pi} M\{f_1(s)\} + \frac{2}{\pi} M\{f_2(t)\}$$

$$+ \frac{4}{\pi^2} |g(1,1;1^-,1^-)|\} ,$$

and $0 \leq c(g) \leq (2/\pi)^4 V(g)$. In particular, if $g(u,v)$ is positively monotonic, then $V(g) = 1$ and $0 \leq c(g) \leq (2/\pi)^4$.

If the summability method corresponds to ordinary convergence, then $g(u,v) = 0$ unless $u = v = 1$. In this case,

$$M\{f(s,t)\} = M\{|\sin s \sin t|\} = \frac{4}{\pi^2}$$

$$M\{f_1(s)\} = M\{|\sin s|\} = \frac{2}{\pi}$$

$$M\{f_2(t)\} = M\{|\sin t|\} = \frac{2}{\pi}$$

so that $c(g) = (2/\pi)^4$. Conversely, if $g(u,v)$ is positively monotonic and $c(g) = (2/\pi)^4$, then each of the terms in the definition of $c(g)$ must contribute a maximal amount, so that $g(1,1;1^-,1^-) = 1$, and the method is ordinary convergence. Hence of all the Hausdorff summability methods with positively monotonic weight functions, $c(g) = (2/\pi)^4$ if, and only if, the method is ordinary convergence.

Now consider the class of all positively monotonic weight functions for which the difference functions

$$g(u,1) - g(u,1^-)$$

$$g(1,v) - g(1^-,v)$$

are continuous, so that $g(u^+,1;u^-,1^-)$ and $g(1,v^+;1^-,v^-)$ are identically zero. Then, taking the remarks (4.27) into account, $0 \leq c(g) \leq 4/\pi^4$, and $c(g) = 4/\pi^4$ if, and only if, the method is of the Euler type. We prove the last assertion. To do this, we introduce two lemmas.

(4.38) Lemma

If $L_m(\epsilon,r)$ denotes the m-th Lebesgue constant for the Euler means of order r, $0 < r < 1$, in the one dimensional case, then

$$L_m(\epsilon,r) = \frac{2}{\pi^2} \log m + o(\log m) , \quad m \to \infty .$$

(4.39) Lemma

If $a_{ij} \geq 0$, and if at least two of the a_{ij}'s are distinct from zero, then

$$M \left\{ \sum_{i,j} a_{ij} \sin su_i \sin tv_j | \right\} < \frac{4}{\pi^2} \sum_{i,j} a_{ij} .$$

The first lemma is due to Livingston [7, p. 312], where we have sacrificed some of the precision that his result permits. The second lemma is the two dimensional analogue of the corresponding result of Lorch and Newman, [8, p. 290], for the one dimensional case. Since the extension of their proof to the two dimensional case requires no modification, we omit the proof of (4.39).

If, now, $g(u,v)$ is a Hausdorff weight function of the Euler type, then $g(u,v) = g_1(u) g_2(v)$, where

$$\begin{aligned} g_i(t) &= 0 , \quad 0 \leq t < r_i \\ &= 1 , \quad r_i \leq t \leq 1 , \end{aligned}$$

where $0 < r_i < 1$, $i = 1,2$. Then

(4.40) $$L(m,n;g) = L_m(\epsilon,r_1)\, L_n(\epsilon,r_2)$$

$$= \frac{4}{\pi^4} \log m \log n + o(\log m \log n)\ , \quad m,n \to \infty\ ,$$

by Livingston's result (4.38). Conversely, if $g(u^+,1;u^-,1^-)$ and $g(1,v^+;1^-,v^-)$ are identically zero, and $c(g) = 4/\pi^4$, then

$$M\{|\sum_{i,j} \sin su_i \sin tv_j\ g(u_i^+,v_j^+;u_i^-,v_j^-)|\} = 4/\pi^2 \ .$$

Hence by (4.39) , $\sum a_{ij} = \sum g(u_i^+,v_j^+;u_i^-,v_j^-) = 1$, and $a_{ij} \neq 0$ for at most one combination (ij) . Hence the method is of Euler type.

In addition, by following the above argument, it is equally easy to show that for the class of all positively monotonic weight functions, with only one of $g(u^+,1;u^-,1^-)$ and $g(1,v^+;1^-,v^-)$ identically zero, $0 \leq c(g) \leq 8/\pi^4$, and $c(g) = 8/\pi^4$ if and only if the method is a product method of an Euler method and ordinary convergence for the one dimensional case.

We summarize the results in a theorem.

(4.41) Theorem 4

Under the hypothesis of Theorem 3,

$$L(m,n;g) = c(g) \log m \log n + o(\log m \log n)\ , \quad m,n \to \infty\ .$$

where $0 \leq c(g) \leq (2/\pi)^4\, V(g)$, and

$$c(g) = \frac{1}{\pi^2} \{M\{f(s,t)\} + \frac{2}{\pi} M\{f_1(s)\} + \frac{2}{\pi} M\{f_2(t)\} + \frac{4}{\pi^2} |g(1,1;1^-,1^-)|\} .$$

For the class of positively monotonic weight functions, $V(g) = 1$, so that $0 \leq c(g) \leq (2/\pi)^4$, and $c(g) = (2/\pi)^4$ if and only if the method is ordinary convergence. If, in addition, one of $g(u^+,1;u^-,1^-)$, or $g(1,v^+;1^-,v^-)$, is identically zero, then $0 \leq c(g) \leq 8/\pi^4$, and $c(g) = 8/\pi^4$ if and only if the method is a product of ordinary convergence and an Euler method. If both $g(u^+,1;u^-,1^-)$ and $g(1,v^+;1^-,v^-)$ are identically zero, then $0 \leq c(g) \leq 4/\pi^4$, and $c(g) = 4/\pi^4$ if and only if the method is of the Euler type.

APPENDIX

§ 1.

In this section, we give an elementary development of the theory of Hausdorff means for double sequences. Proofs are provided only for those theorems for which we have no reference. In general, the theory is a logical extension of the corresponding theory in the one dimensional case.

(A.1) Definition

Let $A = (a_{mnk\ell})$ be a four dimensional matrix, and let $S = (s_{mn})$ be a two dimensional matrix whose elements are the elements of the double sequence $\{s_{mn}\}$. The two dimensional matrix

$$T = AS \quad , \tag{A.2}$$

whose elements are the elements of the double sequence $\{t_{mn}\}$, where

$$t_{mn} = \sum_{k,\ell=o}^{\infty} a_{mnk\ell}\, s_{k\ell} , \qquad m,n = o,1,2,\ldots , \tag{A.3}$$

is meaningful for every m,n, is a transformation of the double sequence $\{s_{mn}\}$. The matrix A is said to transform the double sequence $\{s_{mn}\}$ into the double sequence $\{t_{mn}\}$.

(A.4) Definition

The sequence $\{s_{mn}\}$ is said to be summable by the matrix A to the sum t if (A.3) is meaningful for every m and n, and if

$$\lim_{m,n \to \infty} t_{mn} = t < \infty \ ,$$

where convergence is in the sense of Pringsheim.

(A.5) Definition

The transformation (A.2) is said to be regular if every convergent sequence $\{s_{mn}\}$ is transformed into a convergent sequence $\{t_{mn}\}$, if t_{mn} is meaningful for every m,n, and if

$$\lim_{m,n \to \infty} t_{mn} = \lim_{k,\ell \to \infty} s_{k\ell} \ . \tag{A.6}$$

The transformation is said to be totally regular if, in addition, (A.6) holds whenever $\{s_{mn}\}$ diverges to positive or negative infinity.

(A.7) Theorem

Summability of bounded sequences $\{s_{mn}\}$ by the matrix $A = (a_{mnk\ell})$ is regular if and only if

1. $$\lim_{m,n \to \infty} a_{mnk\ell} = 0 \ , \qquad k,\ell = 0,1,2, \ldots \ .$$

2. $$\lim_{m,n \to \infty} \sum_{k,\ell=0}^{\infty} a_{mnk\ell} = 1$$

3. $$\lim_{m,n \to \infty} \sum_{k=0}^{\infty} |a_{mnk\ell}| = 0 , \quad \ell = 0,1,2,\ldots \ .$$

4. $$\lim_{m,n \to \infty} \sum_{\ell=0}^{\infty} |a_{mnk\ell}| = 0 , \quad k = 0,1,2,\ldots \ .$$

5. $$\sum_{k,\ell=0}^{\infty} |a_{mnk\ell}| < M < \infty , \qquad m,n = 0,1,2,\ldots \ .$$

To our knowledge, this theorem was first proved by Robinson [9].

(A.8) Definition

The matrix $\rho = (\rho_{mnk\ell})$, whose elements are defined by

$$\rho_{mnk\ell} = (-1)^{k+\ell} \binom{m}{k}\binom{n}{\ell} , \qquad k \leq m , \quad \ell \leq n$$

$$= 0 \quad \text{otherwise}$$

is called a difference matrix.

(A.9) Theorem 1

The difference matrix $\rho = (\rho_{mnk\ell})$ is its own inverse: $\rho = \rho^{-1}$.

Proof

Let

$$h_{mnrs} = \sum_{k,\ell=0}^{m,n} \rho_{mnk\ell}\, \rho_{k\ell rs} \quad .$$

Then

$$\sum_{r,s=0}^{\infty} h_{mnrs} x^r y^s = \sum_{r,s=0}^{\infty} \sum_{k,\ell=0}^{\infty} \rho_{mnk\ell}\, \rho_{k\ell rs} x^r y^s$$

$$= \sum_{k,\ell=0}^{\infty} \rho_{mnk\ell} \sum_{r,s=0}^{\infty} \rho_{k\ell rs} x^r y^s$$

$$= \sum_{k,\ell=0}^{\infty} \rho_{mnk\ell} \sum_{r,s=0}^{k,\ell} (-1)^{r+s} \binom{k}{r}\binom{\ell}{s} x^r y^s$$

$$= \sum_{k,\ell=0}^{m,n} (-1)^{k+\ell} \binom{m}{k}\binom{n}{\ell}(1-s)^k (1-y)^\ell$$

$$= \{1-(1-x)\}^m \{1-(1-y)\}^n$$

$$= x^m y^n \quad .$$

Thus $h_{mnrs} = 1$ when $r = m$ and $s = n$, and $h_{mnrs} = 0$ otherwise. This completes the proof.

(A.10) Definition

Let $\{\mu_{mn}\}$ be a given sequence and let $\mu = (\mu_{mnk\ell})$ be a diagonal matrix whose only non-zero entrees are $\mu_{mn} = \mu_{mnmn}$. The

transformation matrix $H = \rho\mu\rho^{-1}$ is called a Hausdorff matrix corresponding to the sequence $\{\mu_{mn}\}$. The sequence $\{s_{mn}\}$ is said to be summable to s in the Hausdorff sense corresponding to the sequence $\{\mu_{mn}\}$, if the $\{t_{mn}\}$, where

$$T = HS \quad ,$$

approaches s as m,n becomes infinite.

(A.11) Remark

It is obvious that Hausdorff matrices are commutative.

(A.12) Example

Let $A = (a_{mnk\ell})$, where

$$a_{mnk\ell} = \frac{1}{m+1} \cdot \frac{1}{n+1} \quad , \quad k \leq m \; , \quad \ell \leq n$$

$$= 0 \quad \text{otherwise} \quad .$$

To show that this is a Hausdorff matrix, let

$$\mu_{mn} = \frac{1}{m+1} \cdot \frac{1}{n+1} \quad .$$

Then

$$\sum_{k,\ell=0}^{\infty} c_{mnk\ell} x^k y^\ell = \sum_{k,\ell=0}^{\infty} \sum_{r,s=0}^{\infty} \rho_{mnrs} \mu_{rs} \rho_{rsk\ell} x^k y^\ell$$

$$= \sum_{r,s=0}^{\infty} \rho_{mnrs}\, \mu_{rs} \sum_{k,\ell=0}^{\infty} \rho_{rsk\ell} x^k y^\ell$$

$$= \sum_{r,s=0}^{\infty} \rho_{mnrs}\, \mu_{rs}\, (1-x)^r\, (1-y)^s$$

$$= \sum_{r,s=0}^{\infty} \rho_{mnrs} \int_{0,0}^{1,1} \mu^r (1-x)^r v^s (1-y)^s dudv$$

$$= \int_{0,0}^{1,1} (1-u+ux)^m (1-v+vy)^n dudv$$

$$= \sum_{k,\ell=0}^{m,n} \binom{m}{k}\binom{n}{\ell} x^k y^\ell \int_{0,0}^{1,1} u^k (1-u)^{m-k} v^\ell (1-v)^{n-\ell} dudv \quad .$$

Hence

$$c_{mnk\ell} = \binom{m}{k}\binom{n}{\ell} \int_{0,0}^{1,1} u^k (1-u)^{m-k} v^\ell (1-v)^{n-\ell} dudv$$

$$= \frac{1}{m+1} \cdot \frac{1}{n+1}\,, \qquad k \leq m\,, \quad \ell \leq n$$

$$= 0 \quad \text{otherwise} \quad .$$

Hence $A = (a_{mnk\ell})$ is a Haudsorff matrix.

(A.13) Theorem 2

A matrix $A = (a_{mnk\ell})$ is a Hausdorff matrix corresponding to the sequence $\{\mu_{mn}\}$ if and only if its elements have the form

$$a_{mnk\ell} = \binom{m}{k}\binom{n}{\ell} \sum_{r,s=o}^{m-k,n-\ell} (-1)^{r+s}\binom{m-k}{r}\binom{n-\ell}{s} \mu_{k+r,\ell+s}$$

Proof

By (A.3),

$$t_{mn} = \sum_{k,\ell=o}^{\infty} a_{mnk\ell}\, s_{k\ell} \quad .$$

If A is a Hausdorff matrix, then $A = \rho\mu\rho^{-1}$, and so

$$t_{mn} = \sum_{r,s=o}^{m,n} (-1)^{r+s}\binom{m}{r}\binom{n}{s} \mu_{rs} \sum_{k,\ell=o}^{r,s} (-1)^{k+\ell}\binom{r}{k}\binom{s}{\ell} s_{k\ell}$$

$$= \sum_{k,\ell=o}^{m,n} s_{k\ell} \sum_{r=k,s=\ell}^{m,n} (-1)^{k+\ell+r+s} \binom{m}{r}\binom{r}{k}\binom{n}{s}\binom{s}{\ell} \mu_{rs}$$

$$= \sum_{k,\ell=o}^{m,n} s_{k\ell} \sum_{r=k,s=\ell}^{m,n} (-1)^{k+\ell+r+s} \binom{m}{k}\binom{m-k}{r-k}\binom{n}{\ell}\binom{n-\ell}{s-\ell} \mu_{k\ell}$$

$$= \sum_{k,\ell=o}^{m,n} \binom{m}{k}\binom{n}{\ell} s_{k\ell} \sum_{r,s=o}^{m-r,n-s} (-1)^{r+s} \binom{m-k}{r}\binom{n-\ell}{s} \mu_{k+r,\ell+s} \quad .$$

Hence

$$a_{mnk\ell} = \binom{m}{k}\binom{n}{\ell} \sum_{r,s=0}^{m-r,n-s} (-1)^{r+s} \binom{m-k}{r}\binom{n-\ell}{s} \mu_{k+r,\ell+s} \quad .$$

(A.14) Theorem

The Hausdorff method of summability corresponding to the sequence $\{u_{mn}\}$ is regular if and only if

$$(A.15) \qquad u_{mn} = \int_{0,0}^{1,1} u^m v^n d^2 g(u,v) \ , \quad m,n = 0,1,2,\ldots \ ,$$

where $g(u,v)$ is of bounded variation in the sense of Hardy-Krause in the cell $[1,1;0,0]$ and

$$g(u,0) = g(u,0^+) = g(0^+,v) = g(0,v) = 0 \ , \quad 0 \leq u,v \leq 1 \ ,$$

$$g(1,1;0,0) = 1 \quad .$$

For a proof of this important result, the reader is referred to Hallenbach [5] and Adams [1] .

(A.16) Remarks

Relative to the sequence $\{u_{mn}\}$, where the elements u_{mn} are defined by (A.15), the Hausdorff method transforms bounded sequences into bounded sequences whenever $g(u,v)$ is of bounded variation. If it is also true that

$$g(u,o) = g(u,o^+) = g(o^+,v) = g(o,v) = k \quad , \quad 0 \le u,v \le 1 \ ,$$

then the method is regular for null sequences. If, in addition, $g(1,1;0,0) = 1$, the method is regular; and if $g(u,v)$ is positively monotonic, the Hausdorff method is totally regular. See Hallenbach [5].

(A.17) Theorem 3

The Hausdorff transform $\{h_{mn}\}$ of the sequence $\{s_{mn}\}$ may be defined by

$$h_{mn} = \sum_{k,\ell=o}^{m,n} \binom{m}{k}\binom{n}{\ell} s_{k\ell} \int_{o,o}^{1,1} u^k(1-u)^{m-k}v^\ell(1-v)^{n-\ell}d^2g(u,v) \quad .$$

Proof

Let $s_{k\ell} = x^k y^\ell$. Then

$$h_{mn} = \sum_{k,\ell=o}^{m,n} (-1)^{k+\ell}\binom{m}{k}\binom{n}{\ell} u_{k\ell} \sum_{r,s=o}^{k,\ell} (-1)^{r+s}\binom{k}{r}\binom{\ell}{s}x^r y^s$$

$$= \sum_{k,\ell=o}^{m,n} (-1)^{k+\ell}\binom{m}{k}\binom{n}{\ell} u_{k\ell}(1-x)^k(1-y)^\ell$$

$$= \sum_{k,\ell=o}^{m,n} (-1)^{k+\ell}\binom{m}{k}\binom{n}{\ell} \int_{o,o}^{1,1}(u-ux)^k(v-vy)^\ell d^2g(u,v)$$

$$= \int_{0,0}^{1,1} \sum_{k,\ell=0}^{m,n} (-1)^{k+\ell}\binom{m}{k}\binom{n}{\ell}(u-ux)^k(v-vy)^\ell d^2g(u,v)$$

$$= \int_{0,0}^{1,1} (1-u+ux)^m(1-v+vy)^n d^2g(u,v)$$

$$= \int_{0,0}^{1,1} \sum_{k,\ell=0}^{m,n} \binom{m}{k}\binom{n}{\ell}(1-u)^{m-k}u^k x^k(1-v)^{n-\ell}v^\ell y^\ell d^2g(u,v)$$

$$= \sum_{k,\ell=0}^{m,n} \binom{m}{k}\binom{n}{\ell}x^k y^\ell \int_{0,0}^{1,1} u^k(1-u)^{m-k}v^\ell(1-v)^{n-\ell}d^2g(u,v)$$

$$= \sum_{k,\ell=0}^{m,n} \binom{m}{k}\binom{n}{\ell}\, s_{k\ell} \int_{0,0}^{1,1} u^k(1-u)^{m-k}v^\ell(1-v)^{n-\ell}d^2g(u,v) ,$$

which proves the theorem.

§ 2.

Throughout this section, we assume that $f(x,y)$ is periodic with period 2π in each variable, and is of bounded variation in the sense of Hardy-Krause in the period cell. We prove that neither the partial sums of the Fourier series nor the Hausdorff means of these sums, display the Gibbs phenomenon at a point of continuity of $f(x,y)$. We begin by stating a theorem, the proof of which is given by Hobson [6, p. 710].

(A.18) Theorem

If $g(x,y)$ is periodic with period 2π in each variable, and is summable in the period cell, then the Fourier series corresponding to it converges at (x,y), interior to the cell $[\pi,\pi;-\pi,-\pi]$, to the value

$$\frac{1}{4}\{g(x^+,y^+) + g(x^+,y^-) + g(x^-,y^+) + g(x^-,y^-)\},$$

provided that the function $\sum g(x\pm s, y\pm t)$ is bounded and can be expressed as the difference of two functions, each of which is monotone non-decreasing (or non-increasing) with respect to both s and t in some cross-neighborhood of the point (x,y).

(A.19) Remarks

By a cross-neighborhood of the point (x,y), we mean the set $\{(s,t)\}$, such that $(s,t) \in [\pi,\pi;-\pi,-\pi]$, and for some $\delta > 0$, at least one of the conditions, $|s-x| \leq \delta$, or $|t-y| \leq \delta$, is satisfied.

In the proof, Hobson states (p. 700) that the function $(st)^{-1} - (\sin s \sin t)^{-1}$ is bounded in the cell $(\pi/2,\pi/2;0,0)$. It is easy to show, however, that this error does not invalidate his result.

A periodic function which is of bounded variation in the sense of Hardy-Krause satisfies the conditions of the theorem in (A.18) everywhere. Hence we have the

(A.20) Corollary

If, in addition to the above assumptions on $f(x,y)$, we assume that it is normalized, then the partial sums of the Fourier series of $f(x,y)$ converge to $f(x,y)$ everywhere. In particular, the partial sums of the Fourier series of $f(x,y)$ converge to $f(x,y)$ at every point of continuity $f(x,y)$.

(A.21) Lemma 1

Let (s,t) be a point of continuity of $f(x,y)$, and let $s_{k,\ell}(f;x,y)$ be the $k\ell$-th partial sum of the Fourier series of $f(x,y)$. Then for every $\epsilon > o$, there exists a positive integer $p = p(\epsilon)$, and an open neighborhood of (s,t), $U = U(\epsilon)$, such that

$$|s_{k,\ell}(f;x,y) - f(x,y)| < \epsilon$$

for all $(x,y) \in U$, and for all $k,\ell > p$.

Proof

Assume that $f(x,y)$ is normalized. Let $\epsilon > o$ be given. By (A.20), choose $p = p(\epsilon)$ large enough so that $|s_{k\ell}(f;s,t) - f(s,t)| < \epsilon/3$ for all $k,\ell > p$. Let

$$U_{k,\ell} = \{(x,y) : |s_{k,\ell}(f;x,y) - f(s,t)| < 2\epsilon/3\} .$$

Since $s_{k,\ell}(f;x,y)$ is a continuous function and (s,t) is fixed, it is clear that for each pair k,ℓ, $U_{k,\ell}$ is an open neighborhood of the

point (s,t) . Hence also for every arbitrary, positive integer q , the set

$$U_p^{p+q} = \bigcap_{k,\ell=p+1}^{p+q} U_{k,\ell}$$

is an open neighborhood of (s,t) . We assert that the set

$$U_p^{\infty} = \bigcap_{k,\ell=p+1}^{\infty} U_{k,\ell}$$

contains an open neighborhood of (s,t) . Assume the contrary. Then there exists a sequence of points, $\{(x_i,y_j)\}$, such that $(x_i,y_j) \to (s,t)$ as $i,j \to \infty$, and

$$|s_{k,\ell}(f;x_i,y_j) - f(s,t)| \geq 2\epsilon/3$$

for arbitrarily large values of k,ℓ . But by (A.20), $s_{k,\ell}(f;x_i,y_j) \to f(x_i,y_j)$ as $k,\ell \to \infty$. Hence at each of this set of points, we must have

$$|s_{k,\ell}(f;x_i,y_j) - f(x_i,y_j)| < \epsilon/6$$

for k,ℓ sufficiently large. But then

$$|f(x_i,y_j) - f(s,t)| > \epsilon/2 \quad ,$$

and this contradicts that $f(x,y)$ is continuous at (s,t) . This proves the assertion.

Now let U_1 be an open neighborhood of (s,t) contained in

U_p^∞ , and by the continuity of $f(x,y)$ at (s,t) choose a neighborhood, U_2 , of (s,t) , such that $|f(x,y) - f(s,t)| < \epsilon/3$ whenever $(x,y) \in U_2$. Then

$$U = U(\epsilon) = U_1 \cap U_2$$

is the required neighborhood, proving the theorem. For if $(x,y) \in U$, then for $k,\ell > p$,

$$\begin{aligned} &|s_{k,\ell}(f;x,y) - f(x,y)| \\ &\quad \leq |s_{k,\ell}(f;x,y) - f(s,t)| + |f(s,t) - f(x,y)| \\ &\quad < 2\epsilon/3 + \epsilon/3 = \epsilon \ . \end{aligned}$$

(A.22) Corollary 1

If D is a closed set of points in the period cell such that $f(x,y)$ is continuous at each point of D , then the Fourier series of $f(x,y)$ converges to $f(x,y)$ uniformly on D .

Proof

Let $\epsilon > 0$ be given. With each point (s,t) of D , associate a neighborhood $U(\epsilon;s,t)$, and a number $p(\epsilon;s,t)$ such that

$$|s_{k,\ell}(f;x,y) - f(x,y)| < \epsilon$$

whenever $k,\ell > p(\epsilon;s,t)$, and $(x,y) \in U(\epsilon;s,t)$. Then the family

$$\{U(\epsilon;s,t) : (s,t) \in D\}$$

is an open cover of the closed, hence compact, set D, and therefore contains a finite subcover,

$$\{U(\epsilon; s_i, t_j) : (s_i, t_j) \in D, \quad i = 1,2,\ldots,m; \quad j = 1,2,\ldots,n\} .$$

Let p be the largest of the numbers

$$\{p(\epsilon; s_i, t_j) : (s_i, t_j) \in D, \quad i = 1,2,\ldots,m; \quad j = 1,2,\ldots,n\} .$$

Then

$$|s_{k,\ell}(f; s,y) - f(x,y)| < \epsilon$$

whenever $(x,y) \in D$, and $k,\ell > p$. This proves the corollary.

(A.23) Corollary 2

The partial sums of the Fourier series of $f(x,y)$ do not exhibit the Gibbs phenomenon at any point of continuity of $f(x,y)$.

The known results from the one dimensional theory, together with Theorem 1 of Chapter 1, the remarks (1.15), and the above corollary now yield

(A.24) Theorem 4

Let $f(x,y)$ be a normalized, periodic function of bounded variation in the sense of Hardy-Krause in the period cell. The partial sums of the Fourier series of $f(x,y)$ will exhibit the Gibbs phenomenon at every point of discontinuity of $f(x,y)$, and only there.

Nest, we examine the regular Hausdorff means of the partial sums of the Fourier series of $f(x,y)$. As an immediate consequence of (A.18), we have

(A.25) Theorem 5

If $g(x,y)$ is periodic with period 2π in each variable, and is summable in the period cell, then the regular Hausdorff means of the partial sums of the Fourier series corresponding to it converge at (x,y) , interior to the cell $[\pi,\pi;-\pi,-\pi]$, to the value

$$\frac{1}{4}\{g(x^+,y^+) + g(x^+,y^-) + g(x^-,y^+) + g(x^-,y^-)\} ,$$

provided that the function $\sum g(x\pm s, y\pm t)$ is bounded and can be expressed as the difference of two functions, each of which is monotone non-decreasing (or non-increasing) with respect to both s and t in some cross-neighborhood of the point (x,y) .

(A.26) Corollary

If, in addition to the earlier assumptions on $f(x,y)$, we assume that it is normalized, then the regular Hausdorff means of the partial sums of the Fourier series of $f(x,y)$ converge to $f(x,y)$ everywhere. In particular, the regular Hausdorff means converge to $f(x,y)$ at every point of continuity of $f(x,y)$.

The corollary follows from the theorem by recalling that the assumptions on $f(x,y)$ are such that it satisfies the conditions of the

theorem, and noting that $f(x,y)$ is normalized.

(A.27) Lemma 2

Let (s,t) be a point of continuity of $f(x,y)$, and let $h_{k,\ell}(f;x,y)$ be the $k\ell$-th (regular) Hausdorff transform of the partial sums of the Fourier series of $f(x,y)$. Then for every $\epsilon > o$, there exists a positive integer $p = p(\epsilon)$, and an open neighborhood of (s,t), $U = U(\epsilon)$, such that

$$|h_{k,\ell}(f;x,y) - f(x,y)| < \epsilon$$

for all $(x,y) \in U$, and for all $k,\ell > p$.

Proof

To prove the lemma, we observe that the $k\ell$-th Hausdorff transform of the partial sums is a continuous function, and then proceed exactly as in the proof of Lemma 1, taking the corollary (A.26) into account.

(A.28) Corollary 1

If D is a closed set of points in the period cell such that $f(x,y)$ is continuous at each point of D, then the regular Hausdorff means of the partial sums of the Fourier series of $f(x,y)$ converge to $f(x,y)$ uniformly on D.

Proof

The proof is again identical to the proof of the Corollary (A.22).

(A.29) Corollary 2

The regular Hausdorff means of the partial sums of the Fourier series of $f(x,y)$ do not exhibit the Gibbs phenomenon at any point of continuity of $f(x,y)$.

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